UNDERSTANDING AND PROVING THE FORMAL DEFINITION OF A LIMIT

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Editor's Note: *Aderonke wrote this paper in the spring of 2020 when she was a senior at Clearview Regional High School. She is currently in her second semester at Yale University in New Haven, Connecticut.*

Introduction

Limits are foundational to understanding calculus but students may have trouble understanding them. This is because limits are unrelated to previous concepts students learned in mathematics and because the mathematical definition varies greatly from the English definition of the word. Exploring and playing with limits helps students better understand the full meaning of limits and set them up for a successful career in advanced mathematics. This paper will focus on dissecting parts of and understanding the Epsilon-Delta (ε - δ) Definition of a Limit while also exploring its different applications in mathematics.

PROBLEM STATEMENT

The Epsilon-Delta (ε - δ) Definition of a Limit was first used by Augustin-Louis Cauchy, formally defined by Bernard Bolzano, and its modern definition was provided by Karl Weierstrass (Grabiner, 1983). It is defined as

Let f(x) be a function defined on an open interval around x_0 ($f(x_0)$ need not be defined). The limit of f(x) as x approaches x_0 is L $\lim_{x \to x0} f(x) = L$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon .$$

x_0 can be any real number.

ORIGINAL RESEARCH

To first understand this mathematical definition, it is essential to understand the components involved.

Limit Notation

As this is the focus of the paper, it is important to note how limits are set up in equations. The format of limits is $\lim_{x \to x0} f(x)$. This is read as the "limit as x approaches x sub-zero of f of x" or "the limit of f of x as x approaches x sub-zero." The number x_0 is a variable that can be used with any number. For example, x_0 can equal 5 ($\lim_{x \to 5} f(x)$), 20 ($\lim_{x \to 20} f(x)$), or 978($\lim_{x \to 978} f(x)$). Of course, different values of x_0 will yield different values of limits. The function f(x) can be replaced with the algebraic expression. If $f(x) = 4x + 3$ then its limit can be written as $\lim_{x \to 5} 4x + 3$. L is the actual value of the limit. In this example, L = 23.

$$\lim_{x \to 5} 4x + 3$$

Substitute x_0 into f(x) to find its limit.

$$\lim_{x \to 5} 4\,(5) + 3$$

Solve.

$$\Rightarrow \; 20 + 3 = 23$$

$$\lim_{x \to 5} 4x + 3 = 23$$

Epsilon & Delta - Why Are They Used?

Epsilon (ε) and delta (δ) are Greek letters and their lowercase version is used as variables in the definition. δ is used as the number of that is added or subtracted from x to get the horizontal range used in the limit while ε is used as the number of that is added or subtracted from the limit. These symbols were chosen arbitrarily as Augustin-Louis Cauchy used them in his equations while trying to solve formulas. His contemporaries who more formally defined the limit - mathematicians Bernard Bolzano and Karl Weierstrass - continued to use the variables (Grabiner, 1983).

Quantifiers and Their Importance

The quantifiers "for every" and "there exists" in the definition are necessary for the proper use of the definition of a limit. Quantifiers are in place so mathematical definitions can apply to an infinite amount of problems. "For every" $\forall$ is used in regards to " $\varepsilon > 0$" and can be interpreted as all real, positive numbers are included in the definition of a limit - regardless of if they are rational, irrational, integers, decimals, etc. This is called the *universal quantifier* since it applies to all of the arguments (any number represented by a variable) that follow the symbol.

Ex. $\forall x \neq 0,\ x \div x = 1$

Reads: For every x that does not equal 0, x divided by x equals 1.

The quantifier "there exists" $\exists$ means there is another number that is real or imaginary that accompanies the preceding statement. This is called the *existential quantifier* because it guarantees that there is something that satisfies the preceding statement.

Ex. $\exists x (x \geq x^2)$

Reads: There exists an x where x is greater than or equal to x^2.

This is true since x=0 and x=0.4 are solutions, along with infinitely many other numbers.

When used together, these form the basis of propositional logic which are statements that can be proven true or false. This is the basis for computer science and mathematical theorems like the definition of a limit.

Ex. $f(x) = x^2+x+1$: $\forall$x, $\exists$ f(x)

Reads: $f(x) = x^2+x+1$ for every x, there exists f(x).

The order of the qualifiers is also important because they create the validity behind the statements. If in a different order, it can result in mathematically illogical and false statements.

With the symbols, the definition of a limit can be rewritten as follows:

Let f(x) be a function defined on an open interval around x_0 ($f(x_0)$ need not be defined). The limit of f(x) as x approaches x_0 is L $\lim_{x \to x0} f(x) = L$, if $\forall$ ε>0 $\exists$ δ>0 such that for all x $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

x_0 can be any real number.

Double Stem Arrow Symbol ⇒

This arrow symbol ⇒ is the material conditional and means "if something (p) is true, then the other thing (q) is also true". In theoretical logic, it is formatted
p ⇒ q. These statements are only false if p is true and q is false. However, p can be false and q can be either true or false when connected by ⇒ to make a logically correct statement.

Absolute Value Signs & Their Importance

The equation states that for all x, $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$. It is imperative to take notice of the absolute value signs in this part of the equation. The absolute value of a number is the distance a number is from 0 and the positive value of a number.

Since both epsilon and delta are positive (ε >0; δ >0) and there are no negative numbers between 0 and another positive number, the difference of $x-x_0$ and/or f(x)-L must be positive in this inequality. Taking the absolute value of the differences is necessary to ensure they fit this requirement so only their magnitude - and not their direction - are considered.

Putting It All Together

Now that we have the background of all the components together, we can start to decode them in the context of the definition of a limit.

This limit ONLY works if the absolute value of the difference of the equation of f(x)-L (the limit) is less than the value of epsilon. This definition can be used to prove the limit is true after given or finding the limit.

This definition is basically saying that as we approach the x_0 within the bounds of some number epsilon, then f(x) will also be approaching the value of the limit that is within the bounds of some number of delta.

Putting it in Practice:

(Full definition recopied for ease of reference.)

Let f(x) be a function defined on an open interval around x_0 ($f(x_0)$ need not be defined). The limit of f(x) as x approaches x_0 is L $\lim_{x\to x0}$ *f(x) = L, if for every* $\varepsilon>0$ *there exists* $\delta>0$ *such that for all x* $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

x_0 can be any real number

Prove $\lim_{x\to 6} x - 5 = 3$.

First, identify the different components and how they compare to the original parts of the equations:

$$x_0 = 6,\ f(x) = x - 5,\ L = 3$$

So it can be written for the definition as so:

$$(0 <)\ |x - 6| < \delta \Rightarrow |x - 5 - 3| < \varepsilon$$

We can combine like terms,

$$|x - 6| < \delta \Rightarrow |x - 8| < \varepsilon$$

and isolate x by moving terms appropriately to do so.

$$|x| < \delta + 6 \text{ and } |x| < \varepsilon + 8$$

With this information, we know that the absolute value of x must be less than $\delta + 6$ and must be less than $\varepsilon + 8$. We can set equal $\delta + 6$ with $\varepsilon + 8$ because of the transitive property of equality - which also includes inequalities.

$$\delta + 6 = \varepsilon + 8$$

This is the relationship between delta and epsilon which will typically arise in limit proofs. For proving $|x - 6| < \delta \Rightarrow |x - 5 - 3| < \varepsilon$ and its limit:

$$Let\,\delta = \varepsilon + 2$$
$$\text{If } |x - 6| < \delta$$

Replace δ with $\varepsilon + 2$.

$$|x - 6| < \varepsilon + 2$$

Subtract 2 from both sides.

$$|x - 8| < \varepsilon$$

Then this equation can be broken down into parts that fit the theorem in:

$$\text{then } |(x - 5) - 3| < \varepsilon$$
$$f(x) - L < \varepsilon$$

Once this has been proven, any number can satisfy x, ε, or δ to make this true. Many of these steps seem repetitive and obvious after doing it. If that is so, then it was done correctly and we have successfully proven a limit to be true. On the first try it may be like "Why did we do this then?". In reversing the process we are proving it true.

Limits to Infinity

While it may initially be thought that limits to infinity are rare or impossible to make out, it is very common to have functions that are defined when going to infinity. Let's take this equation as an example.

$$\lim_{x \to \infty} x^2 + 4x + 4 = \infty$$

Using the Epsilon-Delta Definition of a Limit

We will use the same process as above to prove this limit.

Identify the different components:

$$x_0 = \infty,\ f(x) = x^2 + 4x + 4,\ L = \infty$$

Rewrite the formal definition with the components of this problem:

$$|x - \infty| < \delta \Rightarrow \left|x^2 + 4x + 4 - \infty\right| < \varepsilon$$

In this case, there are no like terms to combine so we can move straight into isolating x.

$$|x| < \delta + \infty \Rightarrow \left|x^2 + 4x + 4\right| < \varepsilon + \infty$$

On the right side, we can factor the equation:

$$|x| < \delta + \infty \Rightarrow |(x+2)(x+2)| < \varepsilon + \infty$$

These are not as nicely equivalent as the other problem because f(x) is quadratic but that is okay. We can still solve this, just with more manipulation.

$$|x| < \delta + \infty \Rightarrow \left|(x+2)^2\right| < \varepsilon + \infty$$

Isolate the absolute value of x on the right side of the double arrow by doing the square root of each side of the inequality.

$$|x| < \delta + \infty \Rightarrow |(x+2)| < \sqrt{\varepsilon + \infty}$$

Then subtract 2 from each side of the inequality.

$$|x| < \delta + \infty \Rightarrow |x| < \sqrt{\varepsilon + \infty} - 2$$

Now we can set equal the absolute values of x to each other,

$$\delta + \infty = \sqrt{\varepsilon + \infty} - 2$$

Now we can establish the relationship between delta and epsilon needed to prove that $\lim_{x \to \infty} x^2 + 4x + 4 = \infty$

Let $\delta = \sqrt{\varepsilon + \infty} - 2 - \infty$

If $|x - \infty| < \delta$

Replace δ with $\sqrt{\varepsilon + \infty} - 2 - \infty$.

$$|x - \infty| < \sqrt{\varepsilon + \infty} - 2 - \infty$$

Note: The infinity under the radical is less than the other infinities. This is because some infinities are larger than other infinities. The other two infinities at this point are equal to each other.

Because they are equal, when we add infinity to both sides, it cancels out.

$$|x| < \sqrt{\varepsilon + \infty} - 2$$

Now we can start the reversing process that will prove the limit true. Add two to both sides of the inequality.

$$|x + 2| < \sqrt{\varepsilon + \infty}$$

Square both sides of the inequality.

$$\left|(x + 2)^2\right| < \varepsilon + \infty$$

Subtract infinity from both sides of the inequality.

$$\left|(x + 2)^2 - \infty\right| < \varepsilon$$

Expand out this equation and then compare it to the theoretical version in the definition of the limit.

$$\text{then } \left|(x^2 + 4x + 4) - \infty\right| < \varepsilon$$
$$f(x) - L < \varepsilon$$

Since this fits the definition, we have successfully proven the limit of an equation as it approaches infinity.

How to Quickly Find Limits to Infinity

This method is used more commonly to find limits to infinity of a function. However, they are not a substitute for going through the process of analyzing the epsilon-delta definition of a limit; doing this work leads to a better understanding of this concept as a whole.

The highest degree terms are used to find limits to infinity quickly. Any equation where the higher degree is in the denominator, the bottom is growing faster than the top which when infinity is plugged in for the variable, the limit will equal 0.

$$\lim_{x\to\infty} \frac{9x^3 + 2x^2 + 27}{5x^4 + 6x + 3} \to \lim_{x\to\infty} \frac{9x^3}{5x^4} \to \lim_{x\to\infty} \frac{9}{5x} \to \frac{9}{5(\infty)} \to \frac{9}{\infty} = 0$$

$$\lim_{x\to\infty} \frac{9x^3+2x^2+27}{5x^4+6x+3} = 0$$

If the higher degree is in the numerator then the limit is equal to infinity which is also referred to as "the limit does not exist" (DNE).

$$\lim_{x\to\infty} \frac{6x^3+5x^2+1}{7x^2} \to \lim_{x\to\infty} \frac{6x^3}{7x^2} \to \lim_{x\to\infty} \frac{6x}{7} \to \frac{6(\infty)}{7} \to \frac{\infty}{7} = \infty$$

$$\lim_{x\to\infty} \frac{6x^3+5x^2+1}{7x^2} = \infty \text{ or DNE}$$

If the degrees are equal on the numerator and denominator then a numerical value will be the limit.

$$\lim_{x\to\infty} \frac{x^2+8}{4x^2} \to \lim_{x\to\infty} \frac{x^2}{4x^2} \to \lim_{x\to\infty} \frac{1}{4}$$

There's no x to input infinity so:

$$\lim_{x\to\infty} \frac{x^2+8}{4x^2} = \frac{1}{4}$$

Limits in Finding the Definition of a Derivate

Now that limits are understood, it is time to use them in the application of other fundamentals of calculus, derivatives.

What are Derivatives?

Derivatives are the rate of change of a function. They can be defined as a function themselves denoted as f'(x) (f prime of x) or they can be defined at any given point (f'(3) = 5). The definition of a derivative is as follows:

Definition of Derivative (as a function of x)

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Definition of Derivative (at point c)

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

(Image Source: AndyMath.com)

In words, the first equation can be described as:

f prime of x (f'(x)) is equal to the limit as the change in x (Δx; delta x) approaches 0 ($\lim\limits_{\Delta x \to 0}$) of f of the quantity of x + change in x (f(x+ Δx)) minus f of x (f(x)) all over the change in x (Δx).

The second equation can be described as:
f prime of some number c is equal to the limit as the change in x approaches 0 ($\lim\limits_{\Delta x \to 0}$) of f of x minus f of c over x minus c.

The top equation is the formal definition of a derivative which is also called the limit definition of a derivative. Sometimes this is written where Δx is represented by h.

If we are to find the derivative of function $f(x) = 5x^2 - 3x + 7$ then we would solve as follows:

Substitute $(x + \Delta x)$ for everywhere there is an x in the original equation. Then subtract the original equation f(x) from that new equation with (x+ Δx). This is part of the derivative equation.

$$f'(x) = \lim_{\Delta x \to 0} \frac{[5(x+\Delta x)^2 - 3(x+\Delta x) + 7] - [5x^2 - 3x + 7]}{\Delta x}$$

Expand the quantities,

$$f'(x) = \lim_{\Delta x \to 0} \frac{[5(x^2 + x\Delta x + \Delta x^2) - 3(x+\Delta x) + 7] - [5x^2 - 3x + 7]}{\Delta x}$$

and distribute the signs and coefficients.

$$f'(x) = \lim_{\Delta x \to 0} \frac{[(5x^2 + 5x\Delta x + 5\Delta x^2) - 3x - 3\Delta x + 7] - 5x^2 + 3x - 7]}{\Delta x}$$

Afterward, combine like terms.

$$f'(x) = \lim_{\Delta x \to 0} \frac{[(5x\Delta x + 5\Delta x^2 + 3\Delta x)]}{\Delta x}$$

Now divide out the Δx on the top and bottom of the equation.

$$f'(x) = \lim_{\Delta x \to 0} 5x + 5\Delta x + 3$$

Since there is still a Δx remaining, then plug in the number that Δx is approaching, which in this case is 0. This will get rid of the limit term in the next step.

$$f'(x) = \lim_{\Delta x \to 0} 5x + 5(0) + 3$$

That will leave us with the answer, which is the derivative!

$$f'(x) = 5x + 3$$

The second equation resembles another essential mathematical equation: the slope equation. It is defined as is $m = \frac{y_2 - y_1}{x_2 - x_1}$, where m is the slope and (x_1, y_1) and (x_2 , y_2) are coordinate points. Because of this, the derivate can be used to find the slope of an equation at a point. There is a special theory called the Mean Value Theorem in which the slope of two points is used to find the derivative of the point in the middle. (That will not be explored in this paper). However, what separates the slope equation and the definition of a derivative at point c is the limit command in the definition of a derivative at point c. Why?

Slope is used for linear functions (lines). For nonlinear functions, the slope of the tangent line that hits the function at a point is equal to the derivative of a function at that point. This relationship is described as

$$m_{tan} = f'(a)$$

where m is the slope of the tangent line, f' is the derivative function, and a is any point.

This equation can alternatively be stated as "a function's derivative measures the slope of the function's tangent line at a point" (Ault, 2017). The limit command is used to find the derivative of the equation. This is why derivatives are used to find slopes at a point but also why they are not necessarily the same thing.

Conclusion

Many readers may struggle to understand this concept on the first try, and that is okay. This is a complicated topic that takes a lot of practice to fully understand. This paper is a starting point for students to comprehend this concept, whether they have calculus experience or not. By dissecting each part of the equation, we get a fuller understanding of each part's importance to the concept as a whole. Combining each part and seeing how they work in example problems is a cumulation of the information presented in this paper. Demonstrating the use of limits through derivatives ties down this concept to something else so students do not feel that it was learned in vain.

Recommendations for Further Research

If given the opportunity, I would have continued to connect these limits to other topics in calculus including infinite series, more topics related to derivatives - like L'Hopital's rule - and integrating. Infinite series are numbers, typically following some sort of pattern, added to each other forever. Exploring infinite series is the main focus of calculus II. L'Hopital's rule is used when an indeterminate form, $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, is the answer after limit approaches the number x (or any variable). When this happens, it is not the actual limit of the equation at that point so one must take the derivative of the numerator and denominator of the equation separately then divide the numerator's derivative by the denominator's derivative. Finally, substitute the number the variable is approaching to get the actual limit. Integrating, or anti-differentiation, is doing the reverse of the derivative. I would also like to find the theoretical way that the formal definition of a limit connects with the limit definition of a derivative.

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