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#### 2017 WMC Annual Conference

**Featuring Dylan Williams**

**WMC Annual Conference**

May 3-5, 2017

Green Lake Conference Center

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**Empowering Mathematical Learning through Assessment**

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**Wisconsin Teacher of Mathematics**

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**WISCONSIN MATHEMATICS COUNCIL, INC.**

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**Wisconsin Mathematics Council, Inc.**

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Dr. Dylan Williams is emeritus professor of educational assessment at University College London. One of the United Kingdom’s leading experts on assessment, Dylan is an experienced international presenter who specializes in introducing educators to the principles and practice of assessment for learning.

**Dylan Williams** will be presenting a day-long session on Wednesday, May 3rd (see info on left) and two keynote sessions on Thursday, May 4th.

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For more information about the 2017 WMC Annual Conference or if you would like to submit a proposal to speak, please visit wismath.org.
Wisconsin Teacher of Mathematics
Fall 2016

The Wisconsin Teacher of Mathematics, the official journal of the Wisconsin Mathematics Council, is published twice a year. Annual WMC membership includes a 1-year subscription to this journal and to the Wisconsin Mathematics Council Newsletter. The Wisconsin Teacher of Mathematics is a forum for the exchange of ideas. Opinions expressed in this journal are those of the authors and may not necessarily reflect those of the Council or editorial staff.

The Wisconsin Teacher of Mathematics welcomes submissions for the Fall 2016 issue. We encourage articles from a broad range of topics related to the teaching and learning of mathematics. In particular, we seek submissions that:

- Present engaging tasks that can be implemented in the Pk–12 classroom.
- Connect mathematics education research and theory to classroom practice.
- Showcase innovative uses of technology in the classroom.
- Focus on work with preservice teachers in the field.
- Discuss current issues or trends in mathematics education.

Other submissions not focusing on these strands are also welcome. If you have questions or wish to submit an article for review, please email Josh Hertel (jhertel@uw lax.edu) or visit the WMC website for more information (http://www.wismath.org/Write-for-our-Journal). The submission deadline for the Fall 2016 issue is September 2, 2016.

Manuscript Submission Guidelines

- Manuscripts may be submitted at any time.
- All manuscripts are subject to a review process.
- Include the author’s name, address, telephone, email, work affiliation and position.
- Manuscripts should be double-spaced and submitted in .doc or .docx format.
- Embed all figures and photos.
- Send an electronic copy of the to jhertel@uw lax.edu.
The National Council of Teachers of Mathematics selected the Wisconsin Teacher of Mathematics to receive the 2013 Outstanding Publication Award. This prestigious award is given annually to recognize the outstanding work of state and local affiliates in producing excellent journals. Judging is based on content, accessibility, and relevance. The WMC editors were recognized at the 2014 NCTM annual meeting.
The focus of the 2016 Wisconsin Mathematics Council Annual Conference is Mathematics in Action. To prepare for this annual event, a colleague and I wanted to know what this meant to middle school students. Here are some of their thought-provoking responses:

- "Math can be easy at times, but at other times, when you don’t understand math, whoa . . . . Putting math into action is hard."
- "To me, it means letters and numbers dancing around. It’s all trying to confuse you, but it’s like . . . I need to solve you!"
- "Like math that you do on an everyday basis . . . but more, because you use it to actually mean something or do something."
- "Math in action . . . a high intense battle going on between two numbers trying to find the better number, then you add the two."
- "Numbers constantly running away from me, because I just don’t get it."
- "Going out into the community and applying math in your everyday activities."

We were excited to note that many of the students’ responses had a familiar theme: Mathematics in action meant engaging in relevant mathematics through productive struggle. This theme is also aligned to the features of high-quality mathematics education documented in the recent National Council of Teachers of Mathematics publication, *Principles to Actions: Ensuring Mathematical Success of All* (2014), which include the implementation of “tasks that promote reasoning and problem solving” and encourage perseverance by supporting “productive struggle in learning mathematics” (p. 10). In this publication, the authors outline several teacher actions that will promote both the implementation of relevant tasks and productive struggle in your own classroom setting. Examples of these actions can also be seen in all the articles in this issue of the *Wisconsin Teacher of Mathematics* journal.

For example, one teacher action includes “selecting tasks that provide multiple entry points through the use of varied tools and representations” (p. 24). This action implies that students should be encouraged to make critical connections to their past experiences as they grapple with key mathematical ideas through the use of appropriate tools and representations. You will find examples of this action in several of the journal articles from this issue, including Zulfiye Zeybek’s article, “Teaching Geometric Shapes to Second Graders,” Dave Ebert’s article, “Engaging our Students in Upper Level Mathematics,” and Tim Deis and Jodean Grunow’s article, “Mathematics Work Stations: Hands-On Learning to Deepen Understanding: Numbers and Operations.” While reading these articles, I encourage you to think about the mathematical tasks that you implement in your own classroom. Do these tasks allow for multiple access points through varied solution strategies? Do these tasks promote a high cognitive demand by fostering mathematical reasoning and problem solving? Do these tasks engage students in discussing key mathematical ideas and concepts?

Another teacher action includes “giving students time to struggle with tasks, and asking questions that scaffold students’ thinking without stepping in to do the work for them” (p. 52). This action is best summarized by another middle school student’s quote: “Mathematics in action means making us do really hard math problems and when we can solve them we know we have worked hard and it makes us feel good.” That is, students should
“feel good” about struggling with mathematics and know that through perseverance and asking purposeful questions, breakthroughs will “emerge from confusion and struggle” (p. 52). You will also find examples of this action in Matthew Chedister and Jessica Shumway’s article, “The Role of Questioning to Develop Conceptual Understanding,” and Adam Paape’s article, “Using Highlighters to Promote Productive Struggle and Deepen Student Understanding.” Again, while investigating these articles, I encourage you to reflect on your own classroom. Do you provide students with the opportunity to really grapple with mathematical ideas and concepts? Do you critically examine how you will support productive struggle through purposeful questions rather than doing the mathematics for them? Do you find yourself “Praising students for their efforts in making sense of mathematical ideas and perseverance in reasoning through problems” (p. 52)?

I hope you enjoy reading this issue of the Wisconsin Teacher of Mathematics journal and reflect on how you engage students in relevant mathematics through productive struggle. In closing, I would like to leave you with a statement from a middle school student (who was submitting a design for a WMC t-shirt): “Some people think mathematics is about getting 100% on tests and about getting all the problems right, but MATH is really all about understanding!” This particular student summarized my definition of Mathematics in Action.
The spring 2016 issue of the Wisconsin Teacher of Mathematics offers a range of resources for both practice and pedagogy. Three of the articles in this issue focus on classroom resources. In the article “Teaching Geometric Shapes to Second Graders,” Zulfiye Zeybek presents tasks designed to engage early elementary students in thinking about geometric shapes. Using the lens of the van Hiele model of geometric thought, Zeybek considers how students might approach each of these tasks and offers implementation suggestions for the classroom. Dave Ebert’s article, “Engaging our Students in Upper Level Mathematics: Vector Voyage,” is the second in his series and showcases a project to engage students in meaningful work with vectors. “Mathematics Work Stations: Hands-On Learning to Deepen Understanding: Numbers and Operations,” by Tim Deis and Jodean Grunow, is the first in a series of pieces focusing on using learning centers as a means of differentiating instruction. In this article, the authors highlight how they developed centers to help students explore topics related to number and quantity.

In “The Role of Questioning to Develop Conceptual Understanding,” Matthew Chedister and Jessica Shumway analyze the questioning strategies of two teachers. They consider how questioning can play a key role in helping teachers assess what students know and build conceptual understanding.

In “Using Highlighters to Promote Productive Struggle and Deepen Student Understanding,” Adam Paape showcases an assessment strategy for engaging students as sense makers in the meaningful review of mathematical problems.

Joshua Hertel
Jenni McCool
Jennifer Kosiak
The Wisconsin Mathematics Education Foundation (WMEF) was founded in 2010 in order to support math education in the State of Wisconsin. We are growing in the amount of donations each year, we are increasing the number of ways we invest in math educators (see below), and we are also expanding in the number of ways that fellow educators can get involved in the organization. Here are some ways that you can become involved in WMEF.

**Become a Member of the Board of Directors**
- The members of the Board of Directors serve a 3-year term.
- The board meets two or three times per year.
- Each director is responsible for committee work and is expected to attend the WMC annual meeting in Green Lake.

**Become the Scholarship Coordinator**
The coordinator serves as the liaison between each scholarship committee, the WMEF Board of Directors, and the WMC administrative firm. The main tasks are to keep track of deadlines for advertisement, applications, committee recommendations, and publicity when the scholarship winners are selected.

**Become a Member of the Scholarship and Grant Committees**
The WMEF awards three scholarships: the Arne Engebretsen Scholarship for high school seniors and the Sister Mary Petronia van Straten and the Ethel Neijar Scholarships for college students entering math education. The WMEF also sponsors grants for K–12 Wisconsin teachers. Members of these committees read applications and make recommendations to the Board of Directors.

**Become the WMEF Webmaster**
The WMEF website, wmefonline.org, needs periodic updating as people donate to our cause, and it is an important source of information and applications for people who want to take advantage of scholarships, grants, and awards that we give. No programming expertise is necessary to manage the website.

**Become a Volunteer for the Green Lake Activities**
WMEF sponsors many activities at the Green Lake conference, including: the Pi Run, the bucket raffle, and the Heads/Tails Event. Volunteers are needed to help with all of these activities.

If you are interested in any of these opportunities, or have questions about any of them, please contact John Janty, WMEF Chairperson, at jjanty@charter.net or 608-238-7830.
Teaching Geometric Shapes to Second Graders

By Zulfiye Zeybek

Have you ever had students who could recognize two-dimensional shapes but could not define them? Have you ever noticed that students do not understand that a square is a rectangle? Geometry is not only an essential ingredient for developing students’ spatial thinking and visualization skills but also for developing their abilities in deductive reasoning and proving. However, studies have documented students’ struggle with understanding basic geometric concepts including two-dimensional shapes and their properties (Clements & Battista, 1992; van Hiele, 1999).

This article describes how second grade students understand and reason about the properties of two-dimensional shapes. It also provides a detailed description of two activities. These two activities were designed by three second grade teachers and a mathematics educator as part of a summer professional development program. This article aims to provide teachers with information on (a) the prior knowledge that students have about two-dimensional shapes and (b) how to match instruction with students’ prior knowledge while supporting their growth in understanding of these concepts.

The van Hiele Model

The van Hiele Model of geometric thought consists of five sequential levels of thinking and reasoning. These levels are labeled visualization, analysis, informal deduction, formal deduction, and rigor, and they describe characteristics of the understanding process. Research has shown that students can achieve levels from visualization to analysis in the early elementary grades and the informal deduction level in the upper elementary or intermediate grades (Clements & Sarama, 2000). This article aims to introduce instructional activities that are responsive to students’ thinking about geometric shapes in the early elementary grades. The activities that are described in this article aim to help students move from the visualization level to the analysis level. To assist the reader in understanding these two levels, a brief description along with examples from an interview follow (see van Hiele, 1999, for further information).

Visualization level. At this initial stage, geometric concepts are viewed as a whole rather than as having components or attributes. Students can recognize and identify two-dimensional shapes by their visual appearance.

Interviewer: Do you know what is on the face of this shape? [presented the yellow pattern block with the face of a hexagon]

Dan & Amy: It is a hexagon!

Interviewer: How do you know it is a hexagon?

Amy: Because, hexagons look like that!

Dan: Because, hexagons have 6 sides and 6 corners.

How Amy and Dan described two-dimensional shapes differed. Amy reasoned on the basis of feature analysis of visual forms. Dan, on the other hand, was able to identify the shape as having components, which is one of the critical characteristics of the analysis level.

Analysis level. At this level, an analysis of geometric concepts begins. Students recognize the properties of two-dimensional shapes, and they understand that all shapes within a class share common properties.

Interviewer: [pointed out Shape B in Figure 1] Look at Shape B, what shape is it?

Students: Square!

Interviewer: I talked to a student a couple of days ago; he said that this shape [Shape B] is a rectangle. Do you agree with him?

Casey: I agree, because rectangles and squares are similar. Um, they are the same except rectangles can be skinner and taller.

Amy: No! Because, they look different!

Interviewer: Are all of these shapes [referred to the shapes in Figure 1] similar in some sense?

Casey: Yes, they all have 4 sides and 4 corners. They are quads! Mr. B. told us that quad means 4!

Amy: A and E are similar, C and G are similar, and D and E are similar [marked the shapes that she thought similar].

Interviewer: Why do you think that they are similar?

Amy: They look like the same!

In this excerpt, Amy rejected the idea of classifying a square as a rectangle based on visual appearance. Casey, on the other hand, pointed out the similarities of the two shapes. Not only was Casey able to recognize that shapes within a group share common properties, but she was also able to realize that shared properties define larger groups (i.e., all shapes with four sides and...
four corners are called quadrilaterals). Although Casey was not able to state interrelationships among shapes explicitly, she accepted the idea of class inclusion—that one shape can be classified as another.

Figure 1. Amy’s grouping of the shapes that are similar.

Teacher-Designed Activities

Having discussed the visualization and analysis levels, one important question remains: How can we use these levels to design instruction that aims to help students move from one level to the next? According to the van Hiele model, instruction is essential for progression from one level to the next. Next, I outline two activities that a group of teachers designed based on their students’ thinking. These activities were created by teachers with a goal of “providing kinds of instructional activities that help students move beyond merely recognizing two-dimensional shapes based on visual appearances (visualization) to understanding the properties of shapes (analysis) and then to realizing the similarities/differences among shapes and classifying one shape as another.” After discussing each activity, I provide suggestions for implementation.

Activity I: How many shapes can you make from a square? According to the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), students are expected to identify and draw shapes based on geometric properties in second grade (MCC.2.G.1). In the activity presented in Figures 2 and 3, the teachers aimed for students to construct several shapes (rectangles, squares, trapezoids, pentagons, hexagons, and triangles) by folding a paper square in several predetermined ways and then stating their reasons for what made a shape. By constructing several shapes from a square, students have opportunities to recognize shapes in different sizes and forms as well as to realize that a shape can be decomposed into other shapes. This activity is a modified version of an activity from Math by All Means by Chris Confer (1994).

Let’s stop here and think about how Amy, Dan, and Casey, who demonstrated different levels of thinking about geometric shapes, would respond to this activity. Amy’s reasoning primarily focused on the visual appearances of geometric shapes. Therefore, she might respond to the prompt “The shape I constructed is a…” as follows: “The shape I constructed is a pentagon because it looks like a pentagon.” Dan, on the other hand, was able to recognize the components of the shapes. Thus, he might say: “The shape I constructed is a pentagon because it has five sides and five corners.” Casey recognized the components of the shapes and realized that the shapes that share common properties belong to the same group. Thus, she might respond by saying: “All have five sides and five corners, so these are all pentagons.”

Implementation suggestions. The following are implementation suggestions for Activity I.

Highlight the use of math vocabulary. It has been shown that students must understand math vocabulary in order to master content and be able to apply it later (Thompson & Rubenstein, 2000). To deliberately develop students’ math vocabulary associated with geometric shapes (e.g., crease, polygons, vertices), it is suggested that teachers use math vocabulary consistently during instruction. During the first activity, students are explicitly asked to state the names of the shapes and their reasons for what makes a shape. This aims to encourage students to recognize and use math vocabulary associated with geometric shapes, such as the names of various geometric shapes and their properties. It should be noted that students might choose to use informal language to describe the shapes or their properties during instruction (for example, using the word corners), so teachers are encouraged to highlight formal language (in that case, using vertices instead of corners).

Foster mathematical discourse. Effective mathematics teaching requires engaging students in discourse to advance their mathematical learning by sharing ideas and learning things from other perspectives (National Council of Teachers of Mathematics, 2014). Even though the recording sheets for Activity I have been specifically designed to ask students to explain their thinking, it is in teachers’ hands how to orchestrate classroom discourse. Using
purposeful questions (i.e., What makes a shape? How do you know that this is a pentagon or hexagon? What are similarities between these two shapes?) is a fruitful way to get students to share their thinking as well as to learn different perspectives. During instruction, it is suggested that teachers let students share their reasoning for constructed shapes and learn from each other in class. As mentioned above, how students reason in this activity might differ. Having an opportunity to hear from their peers might open a window to a new reasoning strategy for students. For instance, Amy might benefit from hearing from Dan or Casey that shapes have certain components besides their appearances that are essential to name a shape. Moreover, it would be beneficial for students to think about similarities and differences among the constructed shapes. For instance, teachers could ask students to compare the shapes that they constructed (i.e., comparing pentagons that are constructed). As a result, students’ attention may shift more towards shared properties such as number of sides or vertices rather than their visual appearances.

Eliminate time spent on nonmathematical activities. Teachers are encouraged to try to eliminate time spent on nonmathematical activities in order to use their time more productively. For instance, in this...
activity, each student should be provided with a paper square that is approximately four inches on each side. It is suggested that teachers cut out these squares before instruction in order to eliminate any class time spent on the nonmathematical activity of cutting.

**Activity II: Sorting shapes into groups.** Opportunities to sort two-dimensional shapes by their geometric properties allow students not only to better identify shapes but also to observe that different shapes can be alike in some way. Comparing two-dimensional shapes helps students focus on specific as well as distinguishing properties of shapes, which are essential to identify shapes (MCC.2.G.1).

In this activity, students will be provided with various shapes (see Figure 4) in different sizes and orientations and asked to make two groups with these shapes. This activity aims to encourage students to think about the similarities and differences among these shapes and to consider whether the shapes can belong to the same group or to different groups.
Let’s think for a moment about how our sample students might perform on this activity. We might expect Amy to put the shapes that look visually similar in one group as she did in Figure 1. However, we might expect Dan and Casey to group all shapes with the same number of sides and vertices in one group. Moreover, we might expect Casey to recognize that specific properties, such as having a certain number of sides, would define the groups (e.g., quadrilaterals or pentagons).

Implementation suggestions. The following are implementation suggestions for Activity II.

Adapt language to reflect relationships between categories of shapes. In addition to highlighting the use of proper language to define or categorize geometric shapes, as described above, it is essential to adapt language so that it highlights the relationships among shapes. Although it is premature to expect students in the early elementary grades to fully comprehend the relationships between shapes, teachers need to be careful about how they classify shapes in order to eliminate misinformation that students would need to unlearn later (e.g., Copley, 2000). Thus, teachers are encouraged to adapt language that allows for growth in understanding of inclusive relationships among shapes (i.e., defining rectangles as special quadrilaterals or defining triangles as special polygons or as three-sided polygons).

Foster mathematical discourse. In Activity II, students are expected to categorize the shapes into different groups. Sorting rules might differ based on students thinking. Therefore, students might benefit from sharing their sorting rules with each other. Although students at earlier levels, such as Amy, might use less complex reasoning and sort the shapes solely based on visual appearance, other students reasoning at higher levels might use more complex sorting rules, such as sorting the shapes according to their properties.
Additionally, teachers are encouraged to guide students in discussing similarities and differences among the shapes during instruction. Asking guiding questions such as “What are the similarities/differences among these shapes?” might allow students to recognize similarities and differences among the shapes and to better identify them.

Eliminate time spent on nonmathematical activities. In this activity, a teacher may choose to cut out the shapes in Figure 4 and give each group (or pair) a bag of cutout shapes to use for the activity. If this is the case, it is suggested that teachers cut out these shapes before instruction in order to eliminate the time spent on nonmathematical cutting activities.

Provide real-life examples. To rouse curiosity and help students realize that classifying, grouping, comparing, and defining—the core of the activities described in this article—are also common in our real lives, I encourage teachers to use examples found in other subject areas (e.g., science) or in everyday life. For instance, to launch Activity II, teachers could start with a sorting activity that is common in science, such as sorting animals based on where they live (e.g., animals that live on land, animals that live in water, and animals that live both on land and in water). After working through this sorting task, teachers could move to the shape sorting activity.

Summary

The release of the National Council of Teachers of Mathematics’ (2006) Focal Points drew renewed attention to the “big ideas” of geometry. This article aimed to share two activities that were designed to help students proceed from recognizing shapes based on their visual appearances (visualization level) to discussing the shapes in terms of properties and making connections between shapes (analysis level). If students have opportunities to view only traditional forms of

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<table>
<thead>
<tr>
<th>Sorting Shapes into Groups</th>
</tr>
</thead>
<tbody>
<tr>
<td>My sorting rule is:...</td>
</tr>
<tr>
<td>...</td>
</tr>
<tr>
<td>How many groups did you form?...</td>
</tr>
</tbody>
</table>

“Can you form three groups with these shapes?”

“Can you form only two groups with the shapes?”

“Can you form only one group with these shapes?”

Modification of the task: Teachers might also provide two pre-sorted groups (i.e. shapes with square corners vs. shapes without square corners, etc.) and ask students to find out the sorting rule that was applied to form the groups.

Figure 5. Recording sheet for Activity II: Sorting shapes into groups.
shapes, they may experience difficulties in identifying nontraditional forms. Activities I and II are specifically designed to (a) provide students with an opportunity to see shapes in different forms, sizes, and orientations and (b) encourage them to shift their attention more onto the properties, which is essential for moving beyond the visualization level. Identifying shapes involves more than looking at their appearance, and I hope that these activities provide teachers with tools to address these big ideas at the early elementary level.

References

Wisconsin Mathematics Council
Distinguished Mathematics Educator Award

The Distinguished Mathematics Educator Award is the most prestigious award that the Wisconsin Mathematics Council bestows. The award recognizes individuals for their exceptional leadership and service to the state’s mathematics education community.

You may download the nomination form at www.wismath.org. Application deadline is January 31, 2017. The award recipient will be honored at the Thursday evening Celebrate WMC event at the WMC Annual Conference.
Engaging Our Students in Upper Level Mathematics: Vector Voyage

By Dave Ebert, Oregon High School

“Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies” (National Council of Teacher of Mathematics [NCTM], 2014, p. 17). The engagement of all students is essential for student learning, and one very effective way to engage all students is through the use of high-quality tasks. In practice, however, utilizing high-quality tasks in the classroom can be very difficult, especially in high-level mathematics courses.

There is abundant research that demonstrates the importance of using high-quality tasks in the classroom; however, there is also a great amount of research that demonstrates the difficulty of implementing these tasks effectively. Principles to Actions: Ensuring Mathematical Success for All (NCTM, 2014) states that:

Research on the use of mathematical tasks over the last two decades has yielded three major findings:

1. Not all tasks provide the same opportunities for student thinking and learning. (Hiebert et al. 1997; Stein et al. 2009)
2. Student learning is greatest in classrooms where the tasks consistently encourage high-level student thinking and reasoning and least in classrooms where the tasks are routinely procedural in nature. (Boaler and Staples 2008; Hiebert and Wearne 1993; Stein and Lane 1996)
3. Tasks with high cognitive demands are the most difficult to implement well and are often transformed into less demanding tasks during instruction. (Stein, Grover, and Henningsen 1996; Stigler and Hiebert 2004). (p. 17)

In her newest book, Mathematical Mindsets (2016), Jo Boaler provides evidence that the use of high-quality tasks in the mathematics classroom is essential in developing students’ mathematical mindsets. She lists six questions (pp. 77–89) that teachers should consider when selecting and implementing tasks to be used with students.

**Question 1. Can you open the task to encourage multiple methods, pathways, and representations?** This may be the most powerful way to transform a procedural problem into a high-level task. Adding a visual component of a task or asking students to make sense of their solutions are great ways to open a task.

**Question 2. Can you make it an inquiry task?** Students need to pursue problems that ask them to come up with an original idea rather than reproduce a prescribed method demonstrated by the teacher. An excellent way to do this is to turn a problem into a puzzle to be solved.

**Question 3. Can you ask the problem before teaching the method?** Asking students to use their intuition to solve a problem before the teacher demonstrates the formal method to be used opens the task to a wide variety of invented solution strategies and gives meaning to the method that is later taught.

**Question 4. Can you add a visual component?** Many of our students are visual learners, and incorporating a visual component to a problem will allow many more students to understand the task. Additionally, when students at any level are stuck, one excellent strategy is to have them draw a picture of what they know or what they are trying to find.

**Question 5. Can you make it and high ceiling?** Low-floor, high-ceiling tasks allow students at many different levels of ability to approach tasks in a variety of ways. One way to create low-floor tasks is to ask students how they interpret a problem. One way to create high-ceiling tasks is to ask students to create their own similar, but more difficult, problems.

**Question 6. Can you add the requirement to convince and reason?** Justification of solutions is essential in the learning and understanding of mathematics. Students should continually be asked “why” throughout their mathematical discourse. Students can justify their solution strategies by attempting to convince themselves, then convince a friend, and then convince a skeptic.

When selecting or creating tasks to be used in the classroom, teachers need to consider these six questions. It is virtually impossible for one task to answer each of these questions; however, every task should answer at least one of these questions. With this in mind, students in upper level mathematics courses such as Algebra 2 and Pre-Calculus need to be actively engaged in their learning through the use of rich mathematical tasks. What follows is an example of a task used within a unit on vectors. What makes this task especially powerful is that it is used prior to any formal instruction on vectors. After the students have completed the task, formal learning about vectors makes much more sense, resulting in greater student learning.

To introduce this task, students are shown two
brief video clips. The first clip is from the movie Despicable Me (available on YouTube at [https://www.youtube.com/watch?v=bOIE0D1Mb18](https://www.youtube.com/watch?v=bOIE0D1Mb18)), which most students are familiar with. This clip gives a perfect definition of a vector: a quantity represented by an arrow with both direction and magnitude. Current students are slightly less familiar with the second video clip (available on YouTube at [https://www.youtube.com/watch?v=5TY5Fp6O5iM](https://www.youtube.com/watch?v=5TY5Fp6O5iM)), a clip from one of the Indiana Jones movies that shows his travels from the United States to Nepal. Essentially, Indiana Jones is traveling by vectors. Each leg of his journey is a quantity represented by an arrow with both direction and magnitude.

Students are then given a copy of a map with an overlaying grid (Figure 1) and told that, like Indiana Jones, they are going to travel the world via vectors. They need to select five locations to visit around the world, and they must return home when they are done. For each leg of their journey, they need to write their vector in component form, find the magnitude of the vector, and determine the direction angle.

Students are told that component form essentially describes the rectangular steps that must be taken when starting at one coordinate and arriving at a second coordinate. Through class discussion, they discover that the magnitude, or length of a vector, can be found by applying the Pythagorean Theorem. They also discover that the direction angle can be determined by using a trigonometric ratio. The class also discusses the importance of having a reference angle so that an angle measuring any number of degrees is the same angle for everyone in the class. Based on their prior work with the unit circle, the students decide to equate east on the map with zero degrees and measure angles counter-clockwise from there.

Students are then given time to work on their vector voyage task (Figure 2). Not only are they determining the component form, magnitude, and direction angle of their vectors, but they must also add their vectors to discover that the sum of all vectors is (0, 0). Careful—this is only true if students’ journeys remain on the surface of the map. Students are also asked to include a picture of the location and a reason for visiting on their map. For many students, this allows them to demonstrate a connection to another subject area or to their own lives.

The wonderful thing about this task is the formal teaching and learning that follows. After completing this project, students already understand the component form, magnitude, and direction angle of a vector. Vector addition follows intuitively, as do other vector operations. For this task, the problem is asked before teaching the method, which leads to greater

Figure 1. Vector voyage map.
For this project you will select five locations to visit around the world. You will travel via vectors to these locations and back home.

For each location you visit, you need to:

- Mark the city on the map
- Outline and color the country you are visiting
- Include a photo of the location
- Give a reason for visiting that location

Six total vectors will be shown on your map, taking you to each of your locations and back to your starting location.

The bottom half of this page needs to be cut out and included as part of the poster.

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![Figure 2. Vector voyage project.](image)
student understanding.

The vector voyage project is a great example of engaging our students in learning through a rich task. It requires high-level thinking from our students, and gives students a visual, hands-on way to explore vectors without relying on direct teacher instruction. Implementing tasks such as this is essential for student learning. All teachers should make it an ongoing goal to search for and implement these kinds of tasks that meaningfully engage students in mathematics.

References

Mathematics Work Stations: Hands-On Learning to Deepen Understanding: Numbers and Operations

By Tim Deis and Jodean Grunow, University of Wisconsin-Platteville

The use of learning centers is a form of differentiated instruction advocated by many educators. Differentiation has been defined as moderating the pace, level, or type of instruction provided in response to the needs of the student. It asks teachers to know their students well so that they can provide instructional experiences and tasks that will enhance learning. The foundation of successful differentiated instruction includes ongoing assessment, group work, and problem solving. It also requires the instructor to recognize the value of the learner who thinks differently. Research has indicated that differentiated instruction can impact student learning (Rock, Gregg, Ellis, & Gable, 2008).

A challenge of using differentiated instruction is adapting it for 30 different students in a classroom. Elementary instructors are typically the teachers who use this method. It is also challenging for middle and high school instructors, who may work with 30 different students, each with varying mathematical skills, in five different classes. The challenge of knowing every student in a classroom well, in addition to instructing, monitoring, and assessing, can be overwhelming for any individual.

As a part of several teacher workshops, Designing and Using Value Added Assessments, STEM Connects, and I^2 STEM, participants explored and implemented a type of differentiated instruction through the use of learning centers. A learning center is a classroom area that contains a collection of activities or materials designed to teach, reinforce, or extend a particular skill or concept (Kaplan, Kaplan, Madsen, & Gould, 1980). The value of a learning center is that it provides the instructor with observable assessment of student work and problem-solving activities that can be experienced in groups. The activities also allow students to learn as they explore through their own experiences.

When would a teacher use mathematics learning centers? If a concept has several facets, each worthy of investigation, the teacher could choose to develop each portion at a center. If a concept is “deep,” various centers could be developed to gradually deepen understanding with each investigation. In that case, the teacher might choose to have a specific rotation. Prior instruction could be reinforced with parallel, connecting, and reflective centers. Because learning centers provide for movement and individual and group problem-solving scenarios, interjecting them into the curriculum can be nonroutine and stimulating.

There are many resources available for elementary teachers to help create centers in the classroom. For example, Debbie Diller’s Math Work Stations: Independent Learning You can Count On, K–2 and Marilyn Burns’s About Teaching Mathematics are good resources. Finding a book on creating centers for high school math content is more difficult.

In what follows, we highlight a few different centers that were developed as a result of the workshops based on the Number and Quantity standards from the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). These centers were created for individuals who had a wide range of mathematical experience. They were used to help each individual explore content based on the level of their understanding and explore a wide range of topics within each standard. The activities may not fit the exact definition of a center because they do not address a specific topic or concept within a standard. The following investigations for Number and Quantity can be (and were) adapted by instructors for use in their classroom.

Number and Quantity

Students in the middle and high school levels have been introduced to most if not all the number operations and properties. Although the type of number system used with these operations and properties may develop over the course of instruction, the basic techniques remain constant. The centers that were used for the Number and Quantity standards focused on the practice of these procedures and used motivational activities to engage the participants.

Games were implemented in the centers. Most of the games that were used and adapted were found on the Internet and have been used for many years. We used many websites. In particular, the National Council of Teachers of Mathematics website had many ideas that we implemented and adapted.

Many teachers are familiar with the Factor Game, the Product Game, and the Fraction Game. We used the premise of the Product Game and modified it to address algebra skills. The reader will see many similarities between the description of the Product Game and the Algebra Product Game.

The Algebra Product Game board consists of a list of factors and a grid of products. Two players compete to get three squares in a row—up and down, across, or diagonally.

The list of expressions at the bottom of the board composes the factor list, and the expressions in the grid
are the products that can be made by multiplying any of the two factors. Two color markers are required, one to mark one player’s products and the other to mark their opponent’s products.

**Goals.** The winner is the first player to place their colored marker on the grid with three squares in a row—up and down, across, or diagonally.

**Procedures.** Player 1 puts a marker on a factor in the factor list. No square on the product grid is marked with Player 1’s color because only one factor has been marked; it takes two factors to make a product.

Player 2 puts the other marker on any number in the factor list (including the same factor marked by Player 1) and then shades or covers the product of the two factors on the product grid.

Player 1 moves either one of the markers to another number and then shades or covers the new product.

Each player, in turn, moves a marker and marks a product. If a product is already marked, the player does not get a mark for that turn.

The game of Sequence, was also adapted to address topics in algebra. It is similar to the Product game but the concept of chance is implemented.

<table>
<thead>
<tr>
<th>1</th>
<th>3</th>
<th>5</th>
<th>5</th>
<th>x</th>
<th>x²</th>
<th>x − 1</th>
<th>x + 1</th>
<th>x − 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>x + 2</td>
<td>x − 3</td>
<td>x + 3</td>
<td>9</td>
<td>15</td>
<td>21</td>
<td>3x</td>
<td>3x²</td>
<td>3x − 3</td>
</tr>
<tr>
<td>3x + 3</td>
<td>3x − 6</td>
<td>3x + 6</td>
<td>3x − 9</td>
<td>3x + 9</td>
<td>25</td>
<td>35</td>
<td>5x</td>
<td>5x²</td>
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<tr>
<td>5x − 5</td>
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<td>5x − 15</td>
<td>5x + 15</td>
<td>49</td>
<td>7x</td>
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</tr>
<tr>
<td>7x − 7</td>
<td>7x + 7</td>
<td>7x − 14</td>
<td>7x + 14</td>
<td>7x − 21</td>
<td>7x + 21</td>
<td>x³</td>
<td>x² − x</td>
<td>x² + x</td>
</tr>
<tr>
<td>x² − 2x</td>
<td>x² + 2x</td>
<td>x² − 3x</td>
<td>x² + 3x</td>
<td>x⁴</td>
<td>x³ − x</td>
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<td>x³ + 2x</td>
</tr>
<tr>
<td>x³ − 3x</td>
<td>x² + 3x</td>
<td>x² − 2x + 1</td>
<td>x² − 1</td>
<td>x² − 3x + 2</td>
<td>x² + x − 2</td>
<td>x² − 4x + 3</td>
<td>x² + 2x − 3</td>
<td>x² + 2x + 1</td>
</tr>
<tr>
<td>x² − x − 2</td>
<td>x² + 3x + 2</td>
<td>x² + 2x − 3</td>
<td>x² + 4x + 3</td>
<td>x² − 4x + 4</td>
<td>x² − 4</td>
<td>x² − 5x + 6</td>
<td>x² + x − 6</td>
<td>x² + 4x + 4</td>
</tr>
<tr>
<td>x² − x − 6</td>
<td>x² + 5x + 6</td>
<td>x² − 6x + 9</td>
<td>x² − 9</td>
<td>x² + 6x + 9</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Factor List:**

| 1 | 3 | 5 | 7 | x | x − 1 | x − 1 | x − 2 | x − 2 | x − 3 | x − 3 | x + 3 |
**Factor Sequence**

The Algebra Sequence Game board consists of a grid of products. Two players compete to get four or more squares in a row—horizontally, vertically, or diagonally.

**Goal.** The first player to have placed their colored marker on the grid with four or more in a row, in a column, or diagonally will win the game.

This game can be modified by having more than two players competing. The authors suggest having no more than four players.

**Procedures.** Each player will have a collection of colored markers to place on the game board. The markers will be a different color from the markers for their opponent.

Players will need to roll a die and interpret the factor assigned to that number.

<table>
<thead>
<tr>
<th>Die Face</th>
<th>Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x$</td>
</tr>
<tr>
<td>2</td>
<td>$y$</td>
</tr>
<tr>
<td>3</td>
<td>$y + 1$</td>
</tr>
<tr>
<td>4</td>
<td>$y + 1$</td>
</tr>
<tr>
<td>5</td>
<td>$x + 2$</td>
</tr>
<tr>
<td>6</td>
<td>$y + 2$</td>
</tr>
</tbody>
</table>

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<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$y$</td>
<td>$x + 1$</td>
<td>$x + 2$</td>
<td>$x + 2$</td>
<td>$y + 2$</td>
</tr>
<tr>
<td>$xy$</td>
<td>$xy$</td>
<td>$xy + x$</td>
<td>$xy + x$</td>
<td>$xy + 2x$</td>
<td>$xy + 2x$</td>
</tr>
<tr>
<td>$xy$</td>
<td>$y^2$</td>
<td>$y^2 + y$</td>
<td>$y^2 + y$</td>
<td>$y^2 + 2y$</td>
<td>$y^2 + 2y$</td>
</tr>
<tr>
<td>$x^2 + x$</td>
<td>$x^2 + 2x$</td>
<td>$x^2 + 2x + 1$</td>
<td>$xy + x + y + 1$</td>
<td>$x^2 + 3x + 2$</td>
<td>$xy + 2x + y + 2$</td>
</tr>
<tr>
<td>$xy + x$</td>
<td>$y^2 + y$</td>
<td>$xy + x + y + 1$</td>
<td>$y^2 + 2y + 1$</td>
<td>$xy + 2y + x + 2$</td>
<td>$y^2 + 2y + 3$</td>
</tr>
<tr>
<td>$x^2 + 2x$</td>
<td>$x^2 + 3x +$</td>
<td>$x^2 + 3x + 2$</td>
<td>$xy + 2y + x + 2$</td>
<td>$xy + 2x + 2y$</td>
<td>$xy + 2x + 2y + 4$</td>
</tr>
<tr>
<td>$y^2 + 2y$</td>
<td>$y^2 + 2y$</td>
<td>$xy + 2x + y + 2$</td>
<td>$y^2 + 2y + 3$</td>
<td>$xy + 2x + 2y + 4$</td>
<td>$y^2 + 4y + 4$</td>
</tr>
</tbody>
</table>
Player 1 rolls the die and will then put the color marker on the game board at any expression that contains the factor that was rolled.

Player 2 will then roll the die and will put the color marker on the game board at any expression that contains the factor that was rolled.

Players will continue to alternate turns until one player has three of his colored markers that are horizontal, vertical, or diagonal.

At any time a marker is placed on the board, the opposing player can challenge their placement by asking the player to justify their answer. If they cannot explain their placement of the marker, they lose that marker on the board.

Another example of a center that implemented the concept of a game was the Tower of Hanoi. In this center, the mathematical practices are addressed. Problem solving, communication, and modeling (as other practices) can be found in the exploration of this game.

The Tower of Hanoi

The Tower of Hanoi consists of three rods, and a number of disks of different sizes, which can slide onto any rod. The puzzle starts with the disks in a neat stack in ascending order of size on one rod with the smallest at the top, thus making a conical shape.

The objective of the puzzle is to move the entire stack to another rod, obeying the following simple rules:

1. Only one disk can be moved at a time.
2. Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack (i.e., a disk can only be moved if it is the uppermost disk on a stack).
3. No disk may be placed on top of a smaller disk.

**Part I.** What is the smallest number of moves required to complete the Tower of Hanoi game with:

<table>
<thead>
<tr>
<th>Number of Disks</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Moves</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Part II.** The Tower of Hanoi can be completed in 31 moves with five discs and 63 moves with six discs. Talk to your team about how you could work out the number of moves needed for the Tower of Hanoi with \( n \) discs. Can you find a recursive equation that models the number of moves required for \( n \) disks? Can you find an explicit equation?

The previous examples are common games that we adapted in our centers. The idea behind these centers is to motivate learners and to provide practice in their computational skills. The following list is provided to give the reader examples of the types of activities used in the centers for this standard to engage participants and allow them to explore and apply their computational skills.


We have found centers to be so successful that we make sure to include one such investigation in each professional development experience. We encourage readers to try these centers in their classrooms, and we conclude with a few remarks shared by teachers following exercises using mathematics learning centers in professional development situations.

*What a wonderful way to extend learning a concept!*

*It is so nice to work together in a small group to tackle challenging problems.*

*I knew that centers worked well in the elementary school; I never thought about using them in the middle school and high school, but they are great—engaging, motivating, challenging, creative!*
Resources For the Creation of the Center Activities


National Council of Teachers of Mathematics’s Navigation Series

References


The Role of Questioning to Develop Conceptual Understanding

By Matthew Chedister, University of Wisconsin-La Crosse, and Jessica Shumway, Utah State University

Rich tasks are critical to the development of conceptual understanding in a mathematics class. However, without whole-class discussion embedded into these tasks, some of their value is lost. Specifically, high-quality classroom discourse plays an essential role in mathematics instruction because it directs learners’ attention to specific aspects of content and develops students’ mathematical explanation and justification abilities (Kazemi, 1998). Orchestrating an effective classroom discussion, however, is not as simple as just asking questions. It requires a teacher to be purposeful in the types of questions that are asked so that the key ideas are made visible and fully explored. An example of how such a discussion might be facilitated is presented below in an activity in which students were learning about the surface area of prisms.

The Task

In the classes of two teachers, Mr. Smith and Mr. Jones, students had been using paper folding to learn about the surface areas of prisms. Specifically, they had been building “open top prisms” (prisms without bases) to develop the idea that the lateral surface area of any prism or cylinder could be found by taking the perimeter of the base and multiplying by the height of the prism. Students referred to this area as “the lateral surface rectangle.” As a final question, Mr. Smith and Mr. Jones gave students the prompt shown in Figure 1.

The connection that the teachers wanted their students to make was that all three prisms have the same perimeter of the base and height, so the lateral surfaces are equal. Therefore, the only difference in total surface area would be dependent on the bases. Further, the teachers hoped that students would draw a connection to an earlier lesson in which students learned that given a fixed perimeter, more square-like or circular figures have a larger area.

Both teachers gave the task to their students to work on for about 10 minutes and then concluded with a whole-class discussion. However, the way each teacher ran the discussion and the results of each discussion differed. In what follows, we outline these differences and highlight some productive practices for structuring whole-class discussion.

Mr. Smith’s Discussion

Mr. Smith focused his discussion on making sure that students had done their calculations correctly by using what are often referred to as probing questions. These are questions that require a short, fact-based response (Krussel, Edwards, & Springer, 2004). To supplement these probing questions, Mr. Smith provided explanations as to why responses were either correct or incorrect. An example of this is when Mr. Smith discussed the area of the missing bases with the class (Figure 2).

During the whole-class discussion, Mr. Smith followed very specific patterns in how he asked

![The following nets show the lateral surface rectangle but not the bases of a triangular prism, a square prism, and a cylinder, respectively. Which one has the greatest surface area? Explain.](image)

*Figure 1. Final prompt given to students.*
questions and presented information. Most of his questions began in the form of “what do” or “what did” and required his students to give him a specific piece of information. After receiving this information, Mr. Smith often added his own commentary to guide students toward the next calculation that he was looking for. At no point did Mr. Smith have students discuss why the answers that he received were correct. At one point, he did ask, “Do people agree?” but this question seemed largely rhetorical and did not lead to any discussion.

Mr. Smith followed this discussion by stating the connection between perimeter and area to the class (Figure 3). Although Mr. Smith's class had done the right calculations, it is still unclear whether his students understood the big ideas of the task—that

| Mr. Smith: | Since we’re comparing the surface area in this case is it necessary for us to go on and multiply this by 2 in this case? No, because we know that this one [points to triangle] is going to be the same size, so all we are really going to care about is when we are solving for it is the fact that which has the largest base. Okay? For the square base prism, the second one, what did you find for the area of that one for the area of the base? | Probing |
| Alex: | 64 | |
| Mr. Smith: | 64 [draws square], so it’s 8 by 8 so the area is 64 units squared [labels square]. And finally we have the cylinder and what do we need to do to find the cylinder in this case [draws circle]? | Probing |
| Alex: | Find the radius. | |
| Mr. Smith: | Right, so we have to find the radius one more time. What did people find for the radius for this particular shape? | Probing |
| Alex: | 16 over pi. | |
| Mr. Smith: | 16 over pi [labels circle]. Do people agree or is there a disagreement here? Okay, so what did we find for the area of the base in this case? Michael what did you say? | Probing |
| Michael: | 81.49, about . . . | |
| Mr. Smith: | Do people agree? Okay [labels circle]. So which one of our three shapes is going to have the largest surface area? | Probing |
| Class: | The cylinder. | |

**Figure 2.** Mr. Smith asking probing questions.

| Mr. Smith: | The cylinder. Now what I was talking to a couple groups about is we want to think about why this happens. And a couple things that we talked about is that one thing we related it back to was our tiling activity we did two weeks ago at the beginning of this unit. If we have a fixed perimeter, which rectangle has the largest area? [Probing] | |
| Ashley: | The one that’s closest to a square. | |
| Mr. Smith: | The one that’s closest to square if we are using tiles. If we are not using tiles then we would want to find the perfect square. So in a similar case here we are looking for the most ideal shape here, which is a circle. | |

**Figure 3.** Mr. Smith connecting perimeter and area.
the area of the bases is all that differentiates the surface areas of the three shapes or that, with a fixed perimeter, circles will have a larger area. Thus, he really had no way of judging the success or failure of the lesson until the summative exam at the end of the unit. At that point, it would be difficult to correct any misconceptions that were revealed. Conversely, Mr. Jones took a different path to ensure that his students had developed conceptual understanding.

**Mr. Jones’ Discussion**

Mr. Jones also saw the value in using probing questions to make sure that his students had done the calculations correctly. However, he also extended his discussion by asking his students *challenging questions* to explain why they had

<table>
<thead>
<tr>
<th>Mr. Jones:</th>
<th>A circle <em>[draws a circle]</em>! And what’s the radius of the circle?</th>
<th>Probing</th>
</tr>
</thead>
<tbody>
<tr>
<td>William:</td>
<td>16/pi.</td>
<td></td>
</tr>
<tr>
<td>Mr. Jones:</td>
<td>16/pi. How did you get that [Victoria]?</td>
<td></td>
</tr>
<tr>
<td>Victoria:</td>
<td>By using…by setting up the circumference <em>[makes circle with hands]</em> and making it equal to 32 so the length of the triangle. So 2 pi R equals 32, then divide it by 2. Pi R equals 16, divided by pi, R equals 16 over pi.</td>
<td>Probing</td>
</tr>
<tr>
<td>Mr. Jones:</td>
<td>How did you know that 32 was the circumference of the cylinder?</td>
<td>Challenging</td>
</tr>
<tr>
<td>Victoria:</td>
<td>Well, in um activity 2, or number 12, we determined that the length of the lateral surface area of the top <em>[points to top of rectangular prism]</em> is the surface area of the circle.</td>
<td></td>
</tr>
<tr>
<td>Mr. Jones:</td>
<td>Okay, yeah, we had already talked about that. What did you get, [Victoria], as the area of this base? This circular base?</td>
<td>Probing</td>
</tr>
<tr>
<td>Victoria:</td>
<td>Um, around 81.4.</td>
<td></td>
</tr>
<tr>
<td>Mr. Jones:</td>
<td>All right <em>[writes on board]</em>. So, what’s the big idea here? Why did I have you guys calculate the area of these bases? What do you notice? [Sage]?</td>
<td>Generalizing</td>
</tr>
<tr>
<td>Sage:</td>
<td>In order to find the area of each of the bases, you have to find out which one has the greatest surface area or the greatest lengths of total surface area.</td>
<td></td>
</tr>
<tr>
<td>Mr. Jones:</td>
<td>Oh, why is that?</td>
<td>Challenging</td>
</tr>
<tr>
<td>Sage:</td>
<td>Because of the prisms that we’re making…the prisms and cylinders that we’re making out of paper, they all have the same lateral surface area? I think…yeah. And so the only differences are the area of the bases.</td>
<td></td>
</tr>
<tr>
<td>Mr. Jones:</td>
<td>Can someone just repeat what [Sage] just said? She just made a really good point. [Martin]?</td>
<td>Revoicing</td>
</tr>
<tr>
<td>Martin:</td>
<td>You only need to find the area of the bases because all of the prisms have the same lateral surfaces rectangle, so that measurement is going to be the same. So by finding the measure of both of the bases or just one of the bases, you can find out which one has the biggest surface area.</td>
<td></td>
</tr>
</tbody>
</table>

*Figure 4. Mr. Jones asking a variety of question types.*
Yael: I was thinking the day we were doing strings, for all of them we have to form the base, we have the same length of string and we form it in 3 different shapes. So it’s cool that even though the perimeter is always going to be the same, the circle is always going to have the greatest area according to like the way we set it up . . .

Mr. Jones: Interesting. What do you guys think about what Yael just said? Do you think she’s correct? [Challenging]

David: It brings you back to what we did . . .

Mr. Jones: Oh yeah, show the class what you did. The back table had a really neat way of talking about or showing what Yael just said.

David: We took the cylinder and put the rectangular prism in it . . . it should fit inside it and there should be the square with little arcs, pieces missing from it, showing that it would have the same perimeter or circumference, but with a bigger area for the circle because the square can fit inside of it.

Figure 5. Discussion in Mr. Jones’ class.

done certain work, generalizing questions to emphasize key points, and revoicing questions to determine if multiple students understood the key points (Krussel, Edwards, & Springer, 2004). An example of this occurred when he was discussing the area of the missing bases (Figure 4).

During the whole-class discussion, Mr. Jones pressed students to give him specific answers by starting with “what do” or “what did” questions. However, he followed these questions that began with how or why with further questions to see if students really understood why their calculations were correct. Further, when he checked for agreement, he did not use a rhetorical question. Instead, he pressed for a specific student to explain the key point in their own words.

Later, when it came to the big connection (circles have the largest area given a fixed perimeter), Mr. Jones allowed his students to guide this discussion (Figure 5).

By focusing his discussion in this way, Mr. Jones was able to establish a much clearer picture of his students’ understanding. He was able to tell if they understood the calculations that they performed or if they had just crunched numbers. In addition, he was able to see if they had made a connection to a previous lesson through multiple explanations. This allowed him to informally assess his students’ understanding and insert any needed remediation or review immediately. Further, by asking students to repeat key ideas, he did not limit his focus to a single student but instead addressed the class at large.

Conclusion
Whole-class discussions are critical to developing conceptual understanding. However, simply talking about mathematical procedures and calculations is often not enough to ensure that this understanding has occurred. A successful classroom discussion requires careful selection of a variety of questions. These questions provide teachers with information about whether students did calculations correctly (probing), understand why calculations were completed (challenging), recognize key ideas (generalizing), and ensure that understanding is not limited to a few students (revoicing). Moreover, when these question types are included, opportunities for the whole class to develop conceptual understanding are greatly increased.

References


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<table>
<thead>
<tr>
<th>Yael:</th>
<th>I was thinking the day we were doing strings, for all of them we have to form the base, we have the same length of string and we form it in 3 different shapes. So it’s cool that even though the perimeter is always going to be the same, the circle is always going to have the greatest area according to like the way we set it up . . .</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mr. Jones:</td>
<td>Interesting. What do you guys think about what Yael just said? Do you think she’s correct? [Challenging]</td>
</tr>
<tr>
<td>David:</td>
<td>It brings you back to what we did . . .</td>
</tr>
<tr>
<td>Mr. Jones:</td>
<td>Oh yeah, show the class what you did. The back table had a really neat way of talking about or showing what Yael just said.</td>
</tr>
<tr>
<td>David:</td>
<td>We took the cylinder and put the rectangular prism in it . . . it should fit inside it and there should be the square with little arcs, pieces missing from it, showing that it would have the same perimeter or circumference, but with a bigger area for the circle because the square can fit inside of it.</td>
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</tbody>
</table>
Using Highlighters to Promote Productive Struggle and Deepen Student Understanding

By Adam Paape, Concordia University

Assessment in mathematics education can take many forms, from the formal to the informal—formative to the summative. Many mathematics teachers find themselves spending a large portion of their time grading assessments with the hope that students will learn from their mistakes. The amount of time spent grading can vary from teacher to teacher, but we all know that the time devoted to grading is a precious commodity. As a result, I am constantly looking for effective and efficient assessment strategies to implement in my mathematics classroom.

This past summer, the Teaching Channel (2015) posted a video that showed Leah Alcala implementing a strategy identified as “Highlighting Mistakes: A Grading Strategy.” This grading strategy focused on the use of a highlighter to denote mistakes that students have made on their assessment. Alcala does not provide any other feedback for the students besides the highlighted mistakes. In the video, she instructed her students to work on correcting their mistakes based on the highlighted areas. Alcala’s method of highlighting mistakes has inspired me to create a revised assessment approach, which I describe in this article.

In the student example found in Figure 1, I was assessing my student on her ability to extend a pattern to a later stage. In a study evaluating the assessment practices of 19 high school math teachers, Senk, Beckmann, and Thompson (1997) determined that an average of only 5% of the test items written by their participants required students to use reasoning skills. In this particular assessment, my intention was to pose questions that encouraged my students to use reasoning or justification. This increased the challenge and engaged my students in the third of the Standards for Mathematical Practice (SMP): “construct viable arguments” (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, p. 6). The students continued to engage

![Figure 1. Student work sample.](image-url)
in SMP 3 when they critiqued the reasoning of other students through their varying solution methods to the problems on the assessment. I was looking for my students to be able to persevere in the problem-solving process.

I purposely selected problems for this assessment that were challenging. During the early days of our semester, I provided my students with the continued message that they could grow through struggling on hard problems in mathematics. I made use of the “Week of Inspirational Math” videos and activities created by Jo Boaler and her team at YouCubed.org to further support a growth mindset within my students (YouCubed, 2015). I did this to prepare my students for the moments of struggle that they would experience during our course.

My Role in the Assessment Process

After I collected all of the completed assessments from my students, I preselected three different colored highlighters to represent varying levels of student mistakes. In this version of the highlighting strategy, I used the following colors to differentiate the levels of student mistakes.

- Orange highlighter indicated that there was one small error. The student made a slight computational mistake that did not make a large impact into the overall problem-solving method.
- Pink highlighter indicated that there was one significant error in logic or computation that led the overall problem-solving solution astray.
- Green highlighter indicated that there was a significant misunderstanding on the part of the student as to what the solving of this problem required. The student would need to completely revise his or her thinking and approach to the problem.

I decided to use three levels of highlighting instead of one level, as seen in the Teaching Channel (2015) video, because this allows the students to have a better understanding of their process. I have also found that students find the variety of colors to be helpful in their revision process. For example, a student might decide to address the orange highlighted work first because there was only a small error in that problem-solving strategy. I have also revised Alcala’s method by including some qualitative feedback throughout the assessment. This particular assessment was the first of the semester in this course, so I felt that it was essential to express to my students that I was curious about their problem-solving methods. Wiliam and Leahy (2015) said the following about feedback: “The only thing that matters with feedback is the reaction of the recipient. That’s it. Feedback—no matter how well designed—that the student does not act upon is a waste of time” (Wiliam & Leahy, 2015, p.107). I felt that the descriptive nature of my feedback, in addition to the highlighted prompts, would help to promote a need for action in my learners through student-to-student discourse in the next step of the highlighting-mistakes strategy. I provided feedback in a way to “value students for their perseverance and effort in reasoning and sense making” (National Council of Teachers of Mathematics, 2014, p. 50). My feedback showed my students that their learning was the central and key component of this assessment experience.

Handing Back the Assessment

In my classroom, I arrange students within collaborative groups made up of three to four learners. At the beginning of the semester, I had each student fill out a survey that I used to analyze his or her dispositions towards mathematics. The collaborative groups that I created were heterogeneous, based on students’ self-reported feelings towards mathematics predicated on their past classroom experiences. I have found these groupings to be highly beneficial in facilitating meaningful classroom discourse centered on mathematical ideas.

As soon as I returned the highlighted assessments to my students, I instructed them to think about the color of the highlighting and the nature of my qualitative feedback. Because each question on the assessment required an element of justification by the students, this next step focused on revisiting their overall solution strategy. Discussions between students began immediately. These discussions were focused on content. This is where another layer of productive struggle occurred. The first layer of struggle took place when the students first engaged the problems on their own, and the second layer of struggle arose as I encouraged the students to compare and contrast solution methods to determine where strategies held true or where they went astray.

It is quite common for students at the same table to have varying solution methods for any given problem,
so I intentionally provided assessment questions that allowed room for different interpretations. This room for ambiguity of approach led to wonderful student-to-student interactions at this point of the highlighting strategy. Students defended their methods for solving a problem in a particular way. Their classmates played the role of a skeptic who needed to be convinced that the method made sense. As Boaler notes, “When students see mathematics as a set of ideas and relationships and their role as one of thinking about the ideas, and making sense of them, they have a mathematical mindset” (Boaler, 2016, p. 34). In this assessment paradigm, the individual learner is the owner of the sense making who is developing a mathematical mindset. Because I provided feedback as a guide, students needed to consider the fine-tuning of their approaches to make them into viable problem-solving strategies. It is common for tables to have congenial arguments centered on mathematics, and this sort of mathematical exchange is the kind of interaction that we all strive to experience in our classrooms. I regularly hear students say, “I did my problem this way” or “I saw the problem another way.” There is a sense of ownership that emerges as the learners talk about their strategies. They express ownership of the mathematics.

After about 10 to 15 minutes of discussion at the collaborative tables, I informed the students that they would need to get their revised work back to me by the next day we meet as a class. I allow this extra time to further promote the idea that learning is the central focus of what we do in math class. Ten minutes tends to be a sufficient amount of time for most students to make revisions. However, for my more deliberate and thorough thinkers, I allow the extra time for revisions so that the necessary struggle and learning can occur.

Assessing Revised Work

Upon receiving all of the revisions from my students, I review all of the new work. Students can earn back points on the assessment through well done and mathematically correct revisions. At the beginning of the next class period, I hand back the assessments with the student revisions. I still do not have any grades written on the assessments to remove the temptation of students to compare scores. This also helps students to focus on any additional qualitative feedback that I may have provided. After this class period, I put the grades in our online grading system to allow students to check their scores. Figure 2 shows one student’s attempt to use a method of adding three onto the previous stage to determine the number of blocks in the next stage.

2. Determine the number of blocks in the 50th stage for the pattern below. Write an explanation to justify your thinking.

![Figure 2. Block problem original.](image-url)
She incorrectly took the number of blocks in the fifth stage and attempted to multiply that by 10 to get the fiftieth stage. Upon reviewing her work, she decided to use a model that connected the stage number to the lengths of the “lines” (her word she assigned to the sections of the blocks) as seen in Figure 3. Figure 4 shows a student who correctly identified a way to communicate the horizontal change in her pattern; however, she misrepresented the vertical change in the rectangular pattern. In Figure 5, she identifies and corrects her misrepresentation of the vertical change, which allowed her to correctly determine the fiftieth
stage for the figure.

**The Benefits of the Highlighting Mistakes Strategy**

This assessment strategy encourages learning to continue after the students have taken the initial assessment. All too often, our strategies for formal assessment of students do not provide a continued learning experience for students after they have taken an assessment. We typically grade the assessment, and we might provide some feedback to the student with the hope that he or she will now understand how to do the work properly. The highlighting mistakes strategy of assessment has a number of layers of student engagement with the mathematical content. The idea of owning their solution methods resonates with my students. They are not seeking out the one way that the teacher wants them to solve the problem. Rather, I engage them at multiple points in the assessment process as they become the center of the learning experience and take on the role of sense makers of the mathematics.

**References**


State Mathematics Competition

The following problem is from the 2015 Middle School State Mathematics Contest. For additional questions and solutions, visit wismath.org/contests.

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Dr. Elizabeth Hughes is an Associate Professor of Mathematics at the University of Northern Iowa. Her research interests include designing practice-based learning experiences for teachers and examining the development of teachers' mathematical knowledge for teaching.

Dr. Nancy C. Anderson is the K-8 Mathematics Coordinator and a Grade 8 Mathematics Teacher at Milton Academy. Nancy is an experienced classroom teacher, curriculum specialist, and professional development leader. She is a frequent speaker at regional and national conferences, and a published author in the field of mathematics education. Nancy’s publications include Classroom Discussions in Math and Good Questions for Math Teaching, Grades 5-8.

More information will be available soon including the preliminary program and how to register! Visit www.wismath.org.
Wisconsin Teacher of Mathematics
Fall 2016

The Wisconsin Teacher of Mathematics, the official journal of the Wisconsin Mathematics Council, is published twice a year. Annual WMC membership includes a 1-year subscription to this journal and to the Wisconsin Mathematics Council Newsletter. The Wisconsin Teacher of Mathematics is a forum for the exchange of ideas. Opinions expressed in this journal are those of the authors and may not necessarily reflect those of the Council or editorial staff.

The Wisconsin Teacher of Mathematics welcomes submissions for the Fall 2016 issue. We encourage articles from a broad range of topics related to the teaching and learning of mathematics. In particular, we seek submissions that:

- Present engaging tasks that can be implemented in the Pk–12 classroom.
- Connect mathematics education research and theory to classroom practice.
- Showcase innovative uses of technology in the classroom.
- Focus on work with preservice teachers in the field.
- Discuss current issues or trends in mathematics education.

Other submissions not focusing on these strands are also welcome. If you have questions or wish to submit an article for review, please email Josh Hertel (jhertel@uwlax.edu) or visit the WMC website for more information (http://www.wismath.org/Write-for-our-Journal). The submission deadline for the Fall 2016 issue is September 2, 2016.

Manuscript Submission Guidelines

- Manuscripts may be submitted at any time.
- All manuscripts are subject to a review process.
- Include the author’s name, address, telephone, email, work affiliation and position.
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For more information about the 2017 WMC Annual Conference or if you would like to submit a proposal to speak, please visit wismath.org.

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