WISCONSIN HIGH SCHOOL STATE MATHEMATICS MEET
WISCONSIN MATHEMATICS COUNCIL
March 1 – 5, 2021

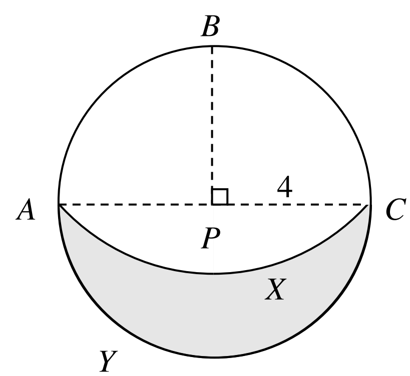
Solutions

Problem set #1

1. After 1 oz. of white paint is put into container B, it is $\frac{6}{7}$ red and $\frac{1}{7}$ white. When 1 oz. is taken from container B, $\frac{6}{7}$ oz. of it is red and $\frac{1}{7}$ oz. of it is white as it is put into container A. At this point, $\frac{6}{7}$ of container B is red, or in terms of ounces, $\frac{36}{7}$ oz. is red and $\frac{6}{7}$ oz. is white. After the 1 oz. mixture is added to container A, there are $\frac{36}{7}$ oz. of white paint and $\frac{6}{7}$ oz. of red paint in container A. Therefore, both containers are equally diluted, so the answer to the question is **neither**.

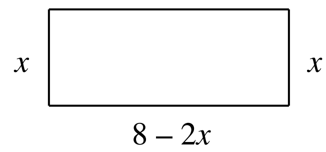
2. Let t = time in seconds for turtle. Then $4t = 48$, or $t = 12$ seconds.
   Let x = time in seconds for entire race. Then $6x = 96$, or $x = 16$ seconds.
   Therefore, the rabbit's time = 4 seconds.
   $r = \frac{d}{t} = \frac{48 \text{ ft}}{4 \text{ sec}} = \boxed{\mathbf{12 \frac{ft}{sec}}}$

3. $\overline{AP}, \overline{BP}$, and $\overline{CP}$ are all radii and congruent, so ΔAPB is an isosceles right triangle, and the lengths of $\overline{AP}$ and $\overline{BP}$ are each 4. Applying the Theorem of Pythagoras to ΔAPB, $AB = \sqrt{32}$. Arc $\widehat{ABC}$ is a semicircle, so $m\angle ABC = 90°$. Using $\sqrt{32}$ as a radius, the area of the sector bounded by $ABCXA$ is $\frac{1}{4}\left[\pi\left(\sqrt{32}\right)^2\right] = 8\pi$. The area of ΔABC is $\frac{1}{2}(8)(4) = 16$. The area of region $APCXA$ is equal to the area of sector $ABCXA$ minus the area of ΔABC, or $8\pi - 16$. The area of semicircle $APCYA$ is $\frac{1}{2}\left[\pi(4)^2\right] = \frac{1}{2}[16\pi] = 8\pi$. The area of the shaded portion is the area of the semicircle minus the area of region $APCXA$, or $8\pi - (8\pi - 16) = \boxed{\mathbf{16}}$.



Problem set #2

1. In the figure to the right, let x represent the length of one side. Then $8 - 2x$ will be the length of an adjacent side. The area is 6, therefore $x(8 - 2x) = 6$ and $x^2 - 4x + 3 = (x - 3)(x - 1) = 0$, which yields $x = 3$ or $x = 1$. The maximum of the perimeter, $16 - 2x$, occurs when $x = 1$, so $16 - 2x = 16 - 2(1) =$ $\boxed{\textbf{14 units}}$.



2. Team A may win the series by winning both of two games, or the last of three or four games after winning one of the others. The six possible sequences of wins are:

$$\text{AA}\left(\frac{1}{4}\right), \text{BAA}\left(\frac{1}{8}\right), \text{ABA}\left(\frac{1}{8}\right), \text{BBAA}\left(\frac{1}{16}\right), \text{BABA}\left(\frac{1}{16}\right), \text{and ABBA}\left(\frac{1}{16}\right).$$

   The probability of A winning the series is the sum of all of these probabilities, which is $\boxed{\mathbf{\frac{11}{16}}}$.

3. Let A be the time it would take John to do the job alone, and B be the time it would take Jim to do the job alone. We write these equations and solve simultaneously:

$$7\left(\frac{1}{A}+\frac{1}{B}\right)+\frac{5}{A}=1 \;\Rightarrow\; \frac{12}{A}+\frac{7}{B}=1$$
$$7\left(\frac{1}{A}+\frac{1}{B}\right)+\frac{7}{B}=1 \;\Rightarrow\; \frac{7}{A}+\frac{14}{B}=1$$

   $\Rightarrow A = 17, B = 23.8$. It takes John $\boxed{\textbf{17 days}}$.

Problem set #3

1. The largest value will occur when the leftmost digit is 9, and the other three digits are in decreasing order from left to right. Since 9421 is not divisible by 4, the next largest possibility is $\boxed{\mathbf{9412}}$.

2. $m\angle ABC = 60^\circ$. ΔBCD is isosceles, with $m\angle BCD = m\angle BCA + m\angle ACD = 60^\circ + 90^\circ = 150^\circ$. This makes $m\angle CBD = 15^\circ$. $m\angle ABD = m\angle ABC - m\angle CBD = 60^\circ - 15^\circ =$ $\boxed{\mathbf{45^o}}$.

3. Using 6 and 9 alone, we can get all $N \geq 6$ where $N = 3k$; using one 20 and the 6s and 9s, we can get all $N \geq 26$ where $N = 3k + 2$; using two 20s and the 6s and 9s we can get all $N \geq 46$ where $N = 3k + 1$. From this we can get any $N \geq 44$. Since $43 = 3(14) + 1$, we would need two 20s, which is impossible. Therefore, the largest non-McNugget number is $\boxed{\mathbf{43}}$.

Problem set #4

1.
$$\begin{aligned}
\sin x + \cos x &= \sqrt{2} \\
(\sin x + \cos x)^2 &= (\sqrt{2})^2 \\
\sin^2 x + 2\sin x \cos x + \cos^2 x &= 2 \\
1 + 2\sin x \cos x &= 2 \\
2\sin x \cos x &= 1 \\
\sin x \cos x &= \boxed{\mathbf{\frac{1}{2}}}
\end{aligned}$$

2. Write three equations with three unknowns, and solve the system of equations for G (grapefruit):

$$\begin{array}{rl} 2A+3G+5P=9 & (1) \\ A+2G+\ \ P=4 & (2) \\ \hline 5A+4G+3P=10 & (3) \end{array} \Rightarrow \begin{array}{rl} 2A+3G+5P=9 & (1) \\ 2A+4G+2P=8 & 2(2) \\ \hline \text{-}G+3P=1 & (a) \end{array} \Rightarrow \begin{array}{rl} 5A+10G+5P=20 & 5(2) \\ 5A+\ \ 4G+3P=10 & (3) \\ \hline 6G+2P=10 & (b) \end{array}$$

We took twice equation (2) and subtracted it from equation (1), then we took 5 times equation (2) and subtracted equation (3) from it, eliminating the apples from the problem.

$$\begin{array}{rl} 18G+6P=30 & 3(b) \\ \text{-}2G+6P=\ \ 2 & 2(a) \\ \hline 20G \qquad\ =28 & (c) \end{array} \Rightarrow G=\frac{28}{20}=1.4$$

Tripling equation (b) and subtracting twice equation (a) from it gives us equation (c), which says that the price of one grapefruit is $\boxed{\textbf{\$1.40}}$.

3. There are multiple ways to solve this problem. One involves the Law of Cosines. Look at the top half of the figure only: $c^2 = a^2 + b^2 - 2ab\cos\frac{\theta}{2}$, or $26 = 34 + 16 - 2\cdot\sqrt{34}\cdot 4\cdot\cos\frac{\theta}{2}$. Simplifying, we get $\cos\frac{\theta}{2} = \frac{3}{\sqrt{34}}$. Square both sides: $\cos^2\frac{\theta}{2} = \frac{9}{34}$, and $\sin^2\frac{\theta}{2} = \frac{25}{34}$. Use the double angle identity to get $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \frac{9}{34} - \frac{25}{34} = -\frac{16}{34} = \boxed{\mathbf{-\frac{8}{17}}}$.

   Another method to get the solution is to look at the top half of the figure, and make a right triangle with the coordinates (4, 5), (7, 10), and (7, 5). This may get you $\cos\frac{\theta}{2} = \frac{3}{\sqrt{34}}$ faster than using the Law of Cosines, depending on how you see an approach to solve the problem. Continue as above to get the answer.

A third method does not even involve using a trig identity formula. Using the Law of Cosines on the same triangle as above with vertices at (4, 5), (7, 10), and (7, 0), you will get $10^2 = \left(\sqrt{34}\right)^2 + \left(\sqrt{34}\right)^2 - 2\cdot\sqrt{34}\cdot\sqrt{34}\cdot\cos\theta$. Solving for θ gets you cos $\theta = \boxed{\mathbf{-\frac{8}{17}}}$.

Problem set #5

1. $$\begin{aligned}\log_b 20^{18} &= 18\log_b 20 \\ &= 18(\log_b 4 + \log_b 5) \\ &= 18(2\log_b 2 + \log_b 5) \\ &= 18(2x + y) \\ &= \boxed{\mathbf{36x + 18y}}\end{aligned}$$

2. The most the pirates could earn is $365 \cdot 7 = 2555$ pieces of gold. Because they only earned 11, they lost 2544 pieces of gold in potential income. Each day they didn't work effectively cost them 12 pieces of gold (they didn't earn 7 and were charged 5). $2544 \div 12 = 212$ days of not working. $365 - 212 =$ $\boxed{\textbf{153 days}}$ of plundering.

   This could also be solved by setting up a system of equations. Let W = the number of days the pirates worked, and N = the number of days they did not work:

   $$\begin{aligned}7W - 5N &= 11 \\ W + \ \ N &= 365\end{aligned}$$

   Solving simultaneously, we get W = 153 and N = 212, the same results as above.

3. The infinite triangles put together will form a parallelogram with two sides of length x (we'll use this as the base) and two sides of length $x\sqrt{2}$. The area of the parallelogram must equal its perimeter, so set up this equation and solve: $x \cdot x = x + x\sqrt{2} + x + x\sqrt{2}$

   $$x^2 = 2x + 2\sqrt{2}x$$
   $$x = 2 + 2\sqrt{2}$$

   Area $= x^2 = (2 + 2\sqrt{2})^2 = \boxed{\mathbf{12 + 8\sqrt{2}}}$

4. The solution is the number of ways to get an even number of heads on 50 flips, multiplied by the probability that each of those ways occurs. This is set up as such:

   $P = \sum_{0}^{50} C(50, n) \cdot \left(\frac{20}{21}\right)^n \cdot \left(\frac{1}{21}\right)^{50-n}$, where n is an even number only. The C function in the summation is the combination function, where $C(n,r)=\frac{n!}{r!(n-r)!}$. This is found in the Binomial Theorem, for those familiar with it. It also shows up in Pascal's Triangle as well, which has ties to the Binomial Theorem. This turns out to be a lengthy calculation, but with technology at your disposal, the process goes quicker. By hand, the summation looks like this:

$$P = C(50,0)\left(\frac{20}{21}\right)^{0}\left(\frac{1}{21}\right)^{50} + C(50,2)\left(\frac{20}{21}\right)^{2}\left(\frac{1}{21}\right)^{48} + C(50,4)\left(\frac{20}{21}\right)^{4}\left(\frac{1}{21}\right)^{46} + \cdots$$
$$+ C(50,48)\left(\frac{20}{21}\right)^{48}\left(\frac{1}{21}\right)^{2} + C(50,50)\left(\frac{20}{21}\right)^{50}\left(\frac{1}{21}\right)^{0}$$

The decimal approximation of this series of probabilities is **0.503**, while the full fraction is

$$\frac{649{,}888{,}834{,}550{,}162{,}490{,}406{,}550{,}256{,}923{,}651{,}939{,}383{,}494{,}983{,}086{,}699{,}171{,}281{,}648{,}490{,}001}{1{,}291{,}114{,}435{,}050{,}719{,}026{,}386{,}456{,}475{,}646{,}628{,}666{,}554{,}089{,}222{,}911{,}187{,}324{,}493{,}837{,}291{,}001}.$$

If a student (or a teacher, for that matter) can derive a simpler, more elegant way to set up this calculation, please inform the WMC writing team. We are interested in hearing about your methods to solve a complex problem such as this one!