WISCONSIN HIGH SCHOOL STATE MATHEMATICS MEET
WISCONSIN MATHEMATICS COUNCIL
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Solutions

Problem set #1

1. $\left(\sqrt{2016}\right)\left(\sqrt[3]{2016}\right)\left(\sqrt[6]{2016}\right)=2016^{\frac{1}{2}}\cdot 2016^{\frac{1}{3}}\cdot 2016^{\frac{1}{6}}=2016^{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}}=2016^{1}=\boxed{\mathbf{2016}}$

2. Since $\sin x=\cos(90^\circ-x)$ and $\sin^2 x=\cos^2(90^\circ-x)$, we can make the following substitutions:

$$\begin{aligned}
&\sin^2(10^\circ)+\sin^2(20^\circ)+\sin^2(30^\circ)+\sin^2(40^\circ)+\sin^2(50^\circ)+\sin^2(60^\circ)+\sin^2(70^\circ)+\sin^2(80^\circ)+\sin^2(90^\circ)\\
&=\sin^2(10^\circ)+\sin^2(20^\circ)+\sin^2(30^\circ)+\sin^2(40^\circ)+\cos^2(40^\circ)+\cos^2(30^\circ)+\cos^2(20^\circ)+\cos^2(10^\circ)+\sin^2(90^\circ)\\
&=\left(\sin^2(10^\circ)+\cos^2(10^\circ)\right)+\left(\sin^2(20^\circ)+\cos^2(20^\circ)\right)+\left(\sin^2(30^\circ)+\cos^2(30^\circ)\right)+\left(\sin^2(40^\circ)+\cos^2(40^\circ)\right)\\
&\quad+\sin^2(90^\circ)\\
&=1+1+1+1+1=\boxed{\mathbf{5}}
\end{aligned}$$

3. A law of exponents and logs says that $\log_{x^n}(b)=\log_x(b^{\frac{1}{n}})$, which becomes $\frac{1}{n}\log_x b$. We can rewrite the original equation as such using this law to get all the bases the same, namely 2:

$$\begin{aligned}
\log_2(\log_{32}x)&=\log_{32}(\log_2 x)\\
\log_2(\log_{2^5}x)&=\log_{2^5}(\log_2 x)\\
\log_2(\log_2 x^{\frac{1}{5}})&=\log_2(\log_2 x)^{\frac{1}{5}}\\
\log_2(\tfrac{1}{5}\log_2 x)&=\tfrac{1}{5}\log_2(\log_2 x)\\
\log_2\tfrac{1}{5}+\log_2(\log_2 x)&=\tfrac{1}{5}\log_2(\log_2 x)\\
\log_2\tfrac{1}{5}&=-\tfrac{4}{5}\log_2(\log_2 x)\\
\log_2 5&=\tfrac{4}{5}\log_2(\log_2 x)\\
5\log_2 5&=4\log_2(\log_2 x)\\
\log_2 3125&=\log_2(\log_2 x)^4\\
\boxed{\mathbf{3125}}&=(\log_2 x)^4
\end{aligned}$$

Problem set #2

1. There are $C(7, 2) = 21$ ways to have a domino with different numbers of dots on each half, plus 7 dominos that have double numbers, for a total of $\boxed{\mathbf{28}}$ distinct dominoes.

   The $C(n, r)$ function, more commonly referred to as the Combination function, is used when you are taking r items from a group of n total items, and is evaluated as $C(n,r)=\dfrac{n!}{r!(n-r)!}$.

   Here, $n = 7$ and $r = 2$: $\dfrac{7!}{2!(7-2)!}=\dfrac{7!}{2!5!}=\dfrac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot5\cdot4\cdot3\cdot2\cdot1}=\dfrac{7\cdot6}{2\cdot1}=\dfrac{42}{2}=21.$

2. Cross multiplying, we get $7x+7yi+xi-y=7+7i$. Regrouping the real and imaginary parts together, we get $7x+7yi+xi-y=7+7i$. Now equate real and imaginary parts to get the equations $7x-y=7$ and $x+7y=7$. Solving these equations simultaneously gives us $x=\dfrac{28}{25}$ and $y=\dfrac{21}{25}$. Therefore, $x+y=\boxed{\dfrac{\mathbf{49}}{\mathbf{25}}}$.

3. Note that $1 + 2 + 3 + 4 + 5 + 6 = 21$. If the probability of rolling any particular number is proprtional to the number rolled, then we have $\mathrm{P}(1)=\dfrac{1}{21}$, $\mathrm{P}(2)=\dfrac{2}{21}$, and so on. In order to roll totals of 10 or more, we have pairs of dice showing (4, 6), (5, 5), (6, 4), (5, 6), (6, 5), and (6, 6) with the following probabilities:

   $$\mathrm{P}(4,6)=\frac{4}{21}\cdot\frac{6}{21}=\frac{24}{441}$$
   $$\mathrm{P}(5,5)=\frac{5}{21}\cdot\frac{5}{21}=\frac{25}{441}$$
   $$\mathrm{P}(6,4)=\frac{6}{21}\cdot\frac{4}{21}=\frac{24}{441}$$
   $$\mathrm{P}(5,6)=\frac{5}{21}\cdot\frac{6}{21}=\frac{30}{441}$$
   $$\mathrm{P}(6,5)=\frac{6}{21}\cdot\frac{5}{21}=\frac{30}{441}$$
   $$\mathrm{P}(6,6)=\frac{6}{21}\cdot\frac{6}{21}=\frac{36}{441}$$

   The sum of these probabilities is $\boxed{\dfrac{\mathbf{169}}{\mathbf{441}}}$.

Problem set #3

1.
$$
\begin{aligned}
\sin x+\cos x&=\sqrt{2}\\
(\sin x+\cos x)^2&=\left(\sqrt{2}\right)^2\\
\sin^2 x+2\sin x\cos x+\cos^2 x&=2\\
1+2\sin x\cos x&=2\\
2\sin x\cos x&=1\\
\sin x\cos x&=\boxed{\mathbf{\tfrac{1}{2}}}
\end{aligned}
$$

2. 10^4 = 10,000 and 20^4 = 160,000, so $ABCDE$ is the fourth power of a two-digit number whose first digit is 1. Since $A + C + E = B + D$ is a sufficient condition for divisibility by 11, $ABCDE = 11^4 = 14641$, so $C = \boxed{\mathbf{6}}$.

3. We claim that AR will be largest when $\overline{AR}\perp\overline{YE}$. Note that all triangles AER will be similar in this problem since $\angle A$ always intercepts arc $\overset{\frown}{YE}$, as does $\angle R$ from a different circle. Since we are trying to determine the largest such triangle, this will occur when $\overline{AE}$ and $\overline{ER}$ are diameters, which leads to $\overline{AR}\perp\overline{YE}$. Using the Pythagorean Theorem:

$$
\begin{aligned}
AR=AY+YR&=\sqrt{40^2-28^2}+\sqrt{32^2-28^2}\\
&=\sqrt{816}+\sqrt{240}\\
&=\boxed{\mathbf{4\sqrt{51}+4\sqrt{15}}}
\end{aligned}
$$

Problem set #4

1. The darker shaded area is the white part of the first graph (to the left of the shaded area inside the 3×3 square), whose total area is 9 cm^2. 9 cm^2 – 4.2 cm^2 = $\boxed{\mathbf{4.8\ cm^2}}$

2. The number of elements in the n^{th} set is n. Therefore, the total number of elements in sets 1 through 34 is $1+2+3+\cdots+34=\dfrac{34(35)}{2}=595$. Therefore, the 35^{th} set contains 35 integers, 596 through 630. The sum of these integers is $\dfrac{35(596+630)}{2}=\boxed{\mathbf{21{,}455}}$.

3. If r is the constant ratio, then $S_{24}=(1+r^8+r^{16})S_8$. Letting $x=r^8$: $40=(1+x+x^2)14$, or $x^2+x+1=\dfrac{20}{7}$, or $x^2+x-\dfrac{13}{7}=0$. Using the quadratic equation to solve this equation for x,

we get $x = \dfrac{-1 \pm \sqrt{1-4(1)(\frac{-13}{7})}}{2(1)} = \dfrac{-1 \pm \sqrt{\frac{59}{7}}}{2}$. Since x must be positive, $x = \dfrac{-1+\sqrt{\frac{59}{7}}}{2} \approx 0.9516$.

This makes $r^8 = 0.9516$. Finally, since $S_8 = (1+r^4)S_4$, we get $S_4 = \dfrac{14}{1+\sqrt{0.9516}} \approx \boxed{7.087}$.

Team Problem set

1. There can be only six elements, since there are only six distinct pairs of sets:

   a belongs to sets A and B
   b belongs to sets A and C
   c belongs to sets A and D
   d belongs to sets B and C
   e belongs to sets B and D
   f belongs to sets C and D

   Each set occurs in three of the pairs, so the number of elements in each set is $\boxed{\mathbf{3}}$.

2. Let us assume without loss of generality that the length of the string is 1. Note the location of the cut as x between 0 and 1. The condition holds if $ax \le 1-x$ or $x \ge a(1-x)$. These inequalities reduce to the following:

   $$\begin{array}{lcl} ax \le 1-x & \text{or} & x \ge a(1-x) \\ ax + x \le 1 & & x \ge a - ax \\ x \le \dfrac{1}{a+1} & & ax + x \ge a \\ & & x \ge \dfrac{a}{a+1} \end{array}$$

   If $0 \le x \le \dfrac{1}{a+1}$, then the length of the string this represents is $\dfrac{1}{a+1}$. If $1 \ge x \ge \dfrac{a}{a+1}$, then this represents the same length as the other case, $\dfrac{1}{a+1}$. Together, they make $\boxed{\dfrac{\mathbf{2}}{\boldsymbol{a}\mathbf{+1}}}$.

3. In this game, you can only score n points by scoring $n-1$ points followed by a 1-point basket, or $n-2$ points followed by a 2-point basket, or $n-3$ points followed by a 3-point basket. So if f_n is the number of ways to score n points, we have $f_n = f_{n-1} + f_{n-2} + f_{n-3}$, with $f_0 = 1$, $f_1 = 1$, and $f_2 = 2$. Therefore, we have

   $$\begin{array}{ll} f_3 = 4 & f_7 = 44 \\ f_4 = 7 & f_8 = 81 \\ f_5 = 13 & f_9 = 149 \\ f_6 = 27 & f_{10} = \boxed{\mathbf{274}} \end{array}$$

4. Because of condition (ii), the most number of non-zero entries in the matrix is 3. There are an infinite number of solutions to this problem, but all of them will fit into one of two categories: either $\boxed{\mathbf{A} = \begin{bmatrix} \mathbf{0} & \boldsymbol{b} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{f} \\ \boldsymbol{g} & \mathbf{0} & \mathbf{0} \end{bmatrix} \textbf{ or } \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \boldsymbol{c} \\ \boldsymbol{d} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{h} & \mathbf{0} \end{bmatrix}}$, where $bfg = 1$ or $cdh = 1$, and each of the three entries are distinct. For example, $b = \frac{1}{2}$, $f = -\frac{1}{4}$, and $g = -8$.

5. $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\left(1-\frac{1}{25}\right)\cdots\left(1-\frac{1}{2025}\right)$

$$= \left(1^2-\left(\frac{1}{2}\right)^2\right)\left(1^2-\left(\frac{1}{3}\right)^2\right)\left(1^2-\left(\frac{1}{4}\right)^2\right)\left(1^2-\left(\frac{1}{5}\right)^2\right)\cdots\left(1^2-\left(\frac{1}{45}\right)^2\right)$$

$$= \left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{5}\right)\cdots\left(1-\frac{1}{45}\right)\left(1+\frac{1}{45}\right)$$

$$= \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(\frac{4}{5}\right)\left(\frac{6}{5}\right)\cdots\left(\frac{44}{45}\right)\left(\frac{46}{45}\right)$$

$$= \left(\frac{1}{2}\right)\left(\frac{46}{45}\right)$$

$$= \boxed{\mathbf{\frac{23}{45}}}$$

6. Take a cross section of the cone. Because the vertex angle is a right angle, the cross section is a 45-45-90 triangle. Whatever the radius of the cone's base is, call it r, the slant height of the cone will be $r\sqrt{2}$. When the cone is sliced and opened up, the radius of the sector will be that slant height $R = r\sqrt{2}$. We know the length of the major arc is the circumference of the cone's base: $C = 2\pi r$. The length of the arc is the partial circumference of the sector, which is the circumference of the circle with radius R times the fraction of the circle it subtends:

$$C_{arc} = 2\pi R \cdot \frac{\theta}{360}$$

$$2\pi r = 2\pi(r\sqrt{2}) \cdot \frac{\theta}{360}$$

$$\frac{360}{\sqrt{2}} = \theta$$

$$\boxed{\mathbf{254.558^{o}}} \approx \theta$$