Representational Competence: A Renewed Focus for Classroom Practice in Mathematics

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Ms. Wagner posed this problem to her fourth grade students: "Sophia purchased 4 booklets of stamps. Each booklet contained 28 stamps. How many total stamps did Sophia purchase?" She first asked the students for ideas on how they could represent or show the situation. She recorded their ideas on the board—use connecting cubes, draw a picture, use grid paper, make a table, draw an array, and make a tape diagram. Ms. Wagner then told the students to represent the problem in two different ways and that one way needed to be a visual or physical representation.

This brief glimpse into Ms. Wagner's classrooms reveals an emphasis on the use of varied representations. She had the students brainstorm representations before they even began to work on the problem, and then she allowed the students to decide for themselves which representations they wanted to use to show the situation. In the past, this teacher usually told, and often showed, her students how they should represent the problem, such as having everyone use grid paper or draw an array in the same exact manner. She decided to make a change in her instruction after reading about using representations in the recently released *Principles to Actions: Ensuring Mathematical Success for All* document by the National Council of Teachers of Mathematics (NCTM, 2014).

Even though the use of representations has long been advocated for in the teaching of mathematics (Bruner, 1966; Greeno, 1987), NCTM recently underscored its importance when they identified it as one of eight effective mathematics teaching practices, which provide high-leverage in supporting student learning. This renewed focus on representations highlights the critical role they play in not only deepening student learning of mathematics, but also in providing students with multiple entry points and access to the study of mathematics. The National Research Council (2001) noted, "Mathematics requires representations. In fact, because of the abstract nature of mathematics, people have access to mathematical ideas only through the representations of those ideas" (p. 94).

This article delves further into this renewed focus on representations in the teaching and learning of mathematics. It begins by examining the characteristics of representational competence to develop in our students. Then it discusses types of representations and important connections between and within different modes of representations. Finally, it presents some recommended actions to strengthen classroom practice in mathematics.

What is Representational Competence?

Representational competence in mathematics is the ability to use representations meaningfully to understand and communicate mathematical ideas and to solve problems. In the literature, this ability is sometimes referred to as "representational flexibility" (Greer, 2009), "representational fluency" (Nathan, Alibali, Masarik, Stephens & Koedinger, 2010), or "representational thinking" (Pape & Tchoshanov, 2001). Regardless of the term, each emphasizes the value of students' ability to work proficiently with varied representations and how that ability supports students' success in learning mathematics. In fact, Collins (2011) challenged the profession to elevate the importance of representations when he suggested that "the teaching of representational competence should lie at the center of classroom practice in math and science" (p. 105).

In considering implications for classroom practice, I suggest the following student abilities as illustrative of proficiency or competence in using representations meaningfully in mathematics. Representational competence includes (1) being able to convey a mathematical idea in various forms (Ellsworth & Sindt, 1994); (2) knowing when and why it is appropriate or valuable to use particular mathematical representations (Marshall, Superfine, & Canty, 2010); (3) being able to translate between and within modes of representations (Lesh, Post, & Behr, 1987); and (4) being able to use representations flexibly to solve problems (Greeno & Hall, 1997).

Modes of Representations

Tripathi (2008) emphasized that using "different representations is like examining the concept through a variety of lenses, with each lens providing a different perspective that makes the picture [concept] richer and deeper" (p. 439). For example, understanding how to represent 4 x 28 as equal groups, as an array, as an area model, as well as with symbols in a table or with equations, all contribute to a more robust understanding of multiplication. The framework shown in Figure 1 highlights five modes of representations—contextual, visual, verbal, physical, and symbolic (NCTM, 2014; adapted from Lesh, Post, & Behr, 1987) and critical connections among representations that are needed for deepening student understanding of mathematics.

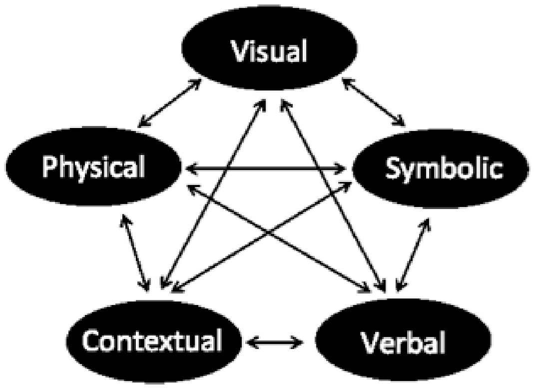

Figure 1. Modes of mathematical representations.

Students need to engage in using and discussing a variety of representations. Figure 2 summarizes some specific representations and student actions for each of the five different modes. Although a specific representation highlights some feature of a particular mathematical idea, at the same time it ignores other aspects. For example, consider the verbal statement, "four times 28" or the use of the terms "factor" and "product." Though this language is important, it is very opaque and conveys little insight into the meaning of multiplication. On the other hand, the verbal statements, "four groups with 28 objects in each group" or "four rows of 28 stamps" or "four jumps of 28" each conveys useful and needed ideas about multiplication as involving equal groups.

Discussion of representations should include asking students to identify similarities and differences among representations. These types of discussions direct attention to essential features of the underlying structure of mathematical ideas and support students' abilities to recognize and utilize those structures in solving problems. Mathematical structure is an important (and often absent) aspect of classroom practice. *The Common Core State Standards for Mathematics* (CCSS-M) identified structure as one of the Standards for Mathematical Practice—"Look for and make use of structure" (NGA/CCSSO, 2010, p. 8). A focus on structure supports students in seeing the coherence of mathematics across content domains within a grade level (e.g., relating multiplication and area in CCSS-M Standards 3.OA.A.3 and 3.MD.C.7 whole numbers and geometry) and the progression of ideas across grades (e.g., using the distributive property from whole numbers to fractions to algebra in CCSS-M Standards 3.OA.B.5, 4.NBT.B.5., 5.NF.B.3, 7.EE.A.1).

Some of the student representations from Ms. Wagner's classroom are shown in Figure 3. What is similar and different about each representation? How does each representation reveal or hide some aspect of the underlying structure of multiplication? Which representations might you select and how might you sequence them for discussion by the whole class to deepen student understanding of multiplication? What questions might you pose for each representation to surface its essential features and make connections among the representations?

Visual Representations	Verbal Representations	Contextual Representations	Physical Representations	Symbolic Representations
Illustrate, show, or work with mathematical ideas using diagrams, pictures, number lines, graphs, and other math drawings.	Use language (words and phrases) to interpret, discuss, define, or describe mathematical ideas, bridging informal and formal mathematical language.	Situate mathematical ideas in everyday, real-world, or imaginary situations, using a variety of discrete and continuous measures (e.g., people, meters, yards).	Use concrete objects to show, study, act upon, or manipulate mathematical ideas (e.g., cubes, counters, tiles, paper strips).	Record or work with mathematical ideas using numerals, variables, tables, and other symbols.

Figure 2. Representational actions.

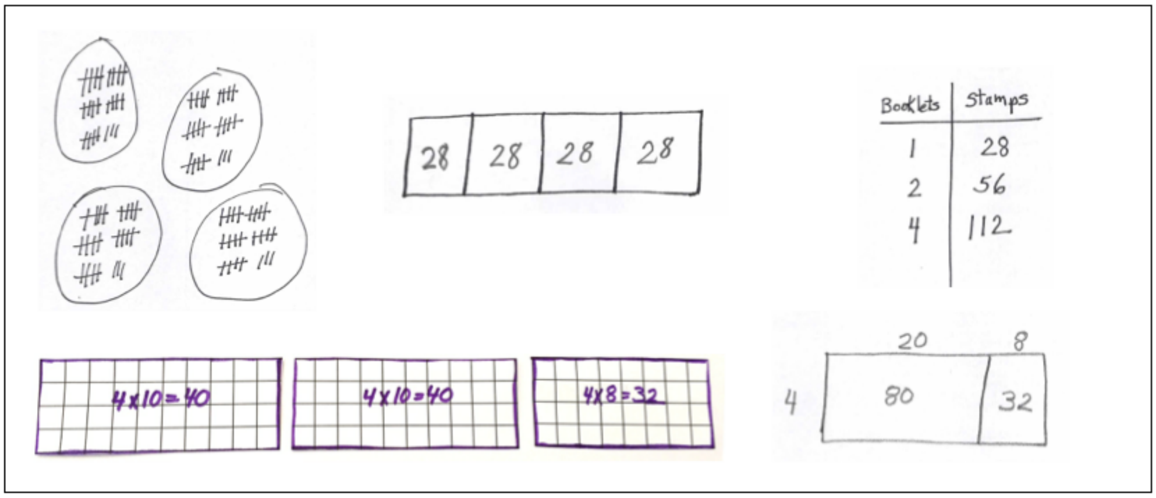

Figure 3. Student representations for the stamp problem.

Translations among Representations

An important aspect of representational competence is the ability to switch or translate from one representation to another. Two important types of translations need to be developed—translations between different modes of representations (e.g., from a visual model to an equation) and translations within a specific mode of representation (e.g., from one visual model to another, such as comparing an array and an area model). The diagram in Figure 4 was modified to show both of these important types of translations.

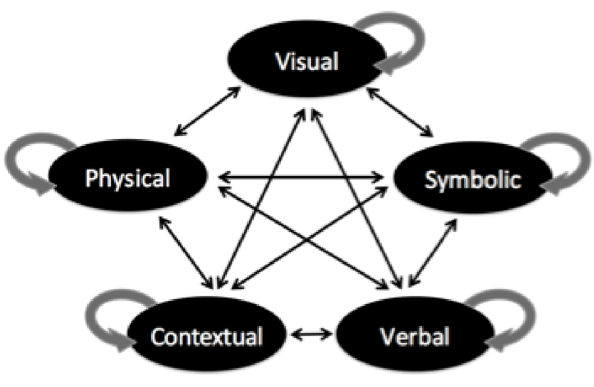

Figure 4. Translations among representations.

Given that students in the United States often perform poorly on international assessments in mathematics (Mullis et al., 1997), Brenner, Herman, Ho, and Zimmer (1999) wondered about the competencies that underlie mathematical performance in high performing countries. They examined the representational skills of sixth grade students in five countries and found students in high performing countries were far more skilled in using and translating among representations than students in the United States. In fact, Chinese and Taiwanese students were nearly five times as likely and Japanese students were nearly three times as likely as U.S. students to correctly respond to items in which they were asked to translate between and within modes of representations (e.g., from a diagram to symbols or from one symbolic expression to an equivalent expression). Cross-national analyses of mathematics curricula and textbooks also show that U.S. students have less systematic exposure to multiple representations than Asian students (Mayer, Sims & Tajika, 1995), and thus likely receive less instruction on becoming skilled in making connections between and within different modes of representations.

Classroom Practice and Representations

Marshall, Superfine, and Canty (2010) suggest three instructional strategies to develop students' representational competence. First, teachers should engage students in dialogue to make explicit the connections among representations. Second, teachers should ask students to alternate or reverse directionality in making connections among representation (e.g., write an equation to match the diagram and then reverse it to draw a diagram that matches an equation). Third, teachers should provide some opportunities for students to decide for themselves which representations to use in solving problems and to ask them to justify or consider the appropriateness (or lack there of) of selected representations (e.g., why did a table work well for this problem or how might a tape diagram also be a good choice).

Ms. Wagner, along with other teachers in her school, has been working toward implementing these recommendations in their classrooms. The following comments summarize some of the shifts in their classroom practice.

- *I've made a big change this year from simply having kids share strategies and representations, to actually having them connect and compare each other's representations.*

- *Before this year, I was not having my students compare and contrast different representations. I now often use the document camera to share multiple representations and ask students what is similar and different about their representations.*

- *Instead of sharing a lot of representations as I did in the past, I am now very selective of the ones I choose and the order in which I share them. I have kids compare and contrast representations more often and even look for efficiency, such as not always using tally marks and trying to use a tape diagram.*

NCTM (2014) renewed our focus on using and connecting representations as an essential component of effective mathematics teaching and learning. They stated, "Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving" (p. 24). Making time for this increased focus on representations is needed. Collins (2011) even suggested, "Mathematics education focuses a lot of time and effort on teaching algorithms, which technological artifacts are able to carry out for them. The time might be better spent in helping students build a strong representational competence" (p. 108).

Ms. Wagner and her colleagues selected to focus on mathematical representations as one of their professional practice goals for the current school year. They have posted the representation diagram shown in Figure 1 in their classrooms and use it as a framework for their interactions with students. In fact, the teachers comment, "We refer to it daily." Furthermore, the teachers noted they use the representation framework for assessing student understanding, identifying gaps in knowledge, and for selecting activities for the whole class, as well as for intervention groups. Finally, they recommend sharing the representation framework with parents during conferences and open-house or curriculum nights to explain the need for increased focus and instructional time on using and connecting mathematical representations.

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