Learning to Ask Questions that Engage Students and Deepen Understanding

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Session Overview

• General discussion of questioning and activities that help us learn more about questioning and instruction
• Sample reflective questions that promote the Mathematical Practices
• Labeling patterns of discussion
• Identifying patterns of discussion in one video clip
• What makes a question “good?”
• Resources
Why Ask Questions?

• To encourage students to participate
• To show we value their thinking
• To inform our teaching decisions
• To help students articulate their thinking
• To encourage student metacognition
• To deepen students’ ability to use the Mathematical Practices
• To help students develop a repertoire of questions to ask themselves.
Activities to Provide Context for Analyzing and Improving Instruction

- Professional readings about general questioning strategies
- Student-centered, problem-based tasks
- Planning a lesson and questions to ask
- Analyzing the effectiveness of questions
- Improving lesson specific questions
Reflective Questions for Noticing Structure

• What am I being asked to do in this problem?

• How would I describe this problem using precise mathematical language?

• Is this problem structured similarly to another problem I’ve seen before?

• What am I trying to solve for?

• What are the relationships between the quantities in the expressions or equations?

Teaching Strategies for Improving Algebra knowledge in Middle and High School Students, April 2015
Reflective Questions for Selecting and Considering Solution Strategies

- What strategies could I use to solve this problem? How many possible strategies are there?

- Of the strategies I know, which seems to best fit this particular problem? Why?

- Is there anything special about this problem that suggests a particular strategy is or is not applicable or a good idea?

- Why did I choose this strategy to solve this problem?

- Could I use another strategy to check my answer? Is that strategy sufficiently different from the one I originally used?
Questions to Facilitate Discussion of Solved Problems

• What were the steps involved in solving the problem? Would they work in a different order?

• Could the problem have been solved with fewer steps?

• Can anyone think of a different way to solve this problem?

• Will this strategy always work? Why or why not?

• How can you change the given problems so that this strategy does not work?

• How can you modify the solution to make it clearer to others?

• What other mathematical ideas connect to this solution?
Classroom Norms That Encourage Engagement in Mathematical Practices

• Every day have a different student read out loud to the class.
• As a teacher, ask a question that students can raise their hand to choose yes or no.
• Do not always call on the same 2-4 students.
• Walk around the classroom while questioning.
• If you notice a ‘shy’ student has the correct answer, then call on him/her to read their answer.
Funneling occurs when teacher asks a series of questions that guide the students through a procedure or to a desired end. In this situation, the teacher is engaged in cognitive activity while the student is merely, answering questions to answer the question.

The teacher takes over the thinking for the students, who may be paying more attention to language cues rather than the mathematical topics at hand.

Herbel-Eisenmann, Beth A., and M. Lynn Breyfogle, Mathematics Teacher, 2005
**Funneling Example**

*Teacher:* A ball has been dropped from rest from the top of a bridge. What is the speed of the ball when it is 5 meters below the drop point?

*Teacher:* Okay, let’s see. What do we know about the acceleration of the ball?

*Students:* It’s 9.8 meters per second squared.

*Teacher:* Good. Now, are we looking for an average speed or an instantaneous speed?

*Students:* Instantaneous. We want to know the speed of the ball when it is 5 meters – no more and no less – below the point of release.

*Teacher:* Precisely! So, how can we find the speed at this point?

*Teacher:* Let’s think about it. What equation can we use that relates instantaneous speed and distance? Anyone?

*Students:* Doesn’t it have something to do with the v-squared equation?

*Teacher:* Yes, $v_{\text{final}}^2 - v_{\text{initial}}^2$ divided by $2ax$ where $a$ is the acceleration and $x$ is the distance.

*Students:* So, solve for $x$; we know that acceleration equals 9.8 meters per second squared.

*Teacher:* You’ve got it!
Questioning our Pattern of Questioning

A focusing-interaction pattern requires the teacher to listen to students’ responses and guide them based on what the students are thinking rather than how the teacher would solve the problem.

Focusing Example

**Teacher:** A ball has been dropped from rest from the top of a bridge. What is the speed of the ball when it is 5 meters below the drop point?

[Long pause – no response from the students.]

**Teacher:** How does one go about solving such a problem? What question do we need to address first?

**Students:** We need to relate the given variables to the unknown.

**Teacher:** Okay, so what are the given variables and what is the unknown?

**Students:** We know that the ball started at rest.

**Teacher:** So what does that tell us?

**Students:** The initial velocity was zero.

**Teacher:** What is the initial acceleration?

**Students:** Zero, it’s not going anywhere to start.

**Teacher:** Hmm, how does one define acceleration?

**Students:** It’s the rate of change of velocity.
**Teacher:** So, if the velocity isn’t changing to start, how can the ball even fall?

**Students:** Oh, yeah, it has to have a non-zero acceleration or it won’t even move.

**Teacher:** Precisely! So, what else do we know?

**Students:** We know the distance – 5 meters.

**Teacher:** What about the 5 meters?

**Students:** It’s the distance that the ball has fallen when we need to find the final velocity.

**Teacher:** Is that the ball’s final velocity? I mean, won’t the ball keep on falling? Maybe the bridge is 15 meters high.

**Students:** We need to know the speed right at 5 meters.

**Teacher:** What else might we call the speed at that point?

**Students:** Instantaneous velocity.

**Teacher:** Good. Now, we have acceleration, initial velocity, and distance of fall. We are looking for instantaneous velocity. Do we need anything else?
Identifying and Modifying Interaction Patterns

Once we identify our current interaction patterns, we can then try to modify them to focus student thinking more often so that students contribute more frequently and can see that we value their thinking.
Surfaces of Revolution - summary

Video clip available at:
http://www.wmich.edu/cpmp/classroomglimpses.html
Viewable via Utube or clips and be downloaded.
Why Ask Focusing Questions?

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- To deepen students’ ability to use the Mathematical Practices
- To help students develop a repertoire of questions to ask themselves.
What Makes a Question a “Good” One?

- It asks students to think about and reflect on the mathematics they learned.

- The teacher should learn something about what the student knows or doesn’t know.

- Students should learn more by answering the question.

- Ones that require students to analyze and defend their thinking and methods.

- Process questions rather than product questions. E.g. Why did you decide to use that approach?

Reinhart, S.C., Never Say Anything a Kid Can’t Say
Resources


• Principles to Action: Ensuring Mathematical Success for All, NCTM (2014)
Resources


Surfaces of Revolution and Cylindrical Surfaces

Graphs of equations in three variables are surfaces. Some of these surfaces can be generated by rotating (or revolving) a curve about a line, sweeping out a surface of revolution. The line about which the curve is rotated is called the axis of rotation. A table leg or lamp base turned on a lathe has a surface of revolution. A potter using a potting wheel makes surfaces of revolution.

Some common surfaces can be thought of as surfaces of revolution. As you work on the problems of this investigation, look for answers to this question:

*What strategies can be used to identify and sketch surfaces of revolution and cylindrical surfaces?*

1 Sketch a graph of \( x^2 + y^2 = 25 \) in the \( xy \)-plane of a three-dimensional coordinate system.

a. Imagine rotating the circle about the \( y \)-axis. What kind of surface is formed?

b. Would you get the same surface if the circle was rotated about the \( x \)-axis? Explain your reasoning.

c. Write the equation for this surface of revolution.

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1 Copyright 2015 McGraw-Hill Education – Permission granted.
The paraboloid from Lesson 1 Problem 5 (page 404) is reproduced in Diagram I. Write an equation of a curve in the $yz$-plane that would produce a similar surface when rotated about an axis. What is the axis of rotation?

Describe, and illustrate with a sketch, how a cone with vertex at the origin (Diagram II) can be generated as a surface of revolution from a two-dimensional figure.

Next consider the segment determined by the points $P(0, 5, 1)$ and $Q(0, 5, 6)$.

a. Imagine rotating the segment about the $z$-axis. What kind of surface is formed? Draw a sketch of the surface.

b. If $A(x, y, z)$ is a point on the surface of revolution, what conditions must be satisfied by $x$, $y$, and $z$?

c. Develop an equation for this surface of revolution.

In the next four problems, you will examine surfaces formed by other familiar curves rotated about the $x$, $y$, or $z$-axis.

Consider the curve $y - x^2 = 0$ in the $xy$-plane. Imagine the curve revolving about the $y$-axis to generate a surface.

a. Describe the traces of this surface.

b. Describe the cross sections parallel to the $xz$-plane.

c. Sketch the surface.

Next consider the curve $z - y^4 = 0$ in the $yz$-plane.

a. Sketch the surface generated by rotating this curve about the $z$-axis.

b. Describe the cross sections formed by planes parallel to the $xy$-plane and give reasons for your answers.
Look back at your work for Problem 2.

a. Use algebraic and geometric reasoning to develop a possible equation for that paraboloid.

b. Based on your work in Part a, what can you conclude about the equation for the surface in Problem 5?

Now consider a line through the origin in the $yz$-plane that makes an angle of $\theta$, $0^\circ < \theta < 90^\circ$, with the positive $z$-axis. Generate a surface by rotating the line about the $z$-axis.

a. Make a sketch of the line and the surface generated. Describe the surface.

b. Describe the cross section of the intersection of the surface and a plane perpendicular to the $z$-axis, other than the $xy$-plane.

c. To find the equation of this surface, you can use reasoning similar to that used in Problems 6 and 7. Let $P(x, y, z)$ be any point on the surface.

i. Explain why $x^2 + y^2 = r^2$, where $r$ is a function of $z$.

ii. Explain why $r = |z| \tan \theta$.

iii. Explain why $x^2 + y^2 = k^2z^2$, where $k^2 = \tan^2 \theta$.

d. What are the equations of the traces of this surface?

e. Use the equation of the surface to determine the shapes of cross sections parallel to the $yz$- and $xz$-planes.

In Problem 4, you formed a cylinder by rotating a segment perpendicular to the $xy$-plane about the $z$ axis. More generally, a cylindrical surface can be generated by moving a line along the path of a plane curve keeping the line at a fixed angle to the curve so that as the line moves, it is always parallel to its original position. The horizontal cross sections will all be congruent, but do not need to be circles as in pipes and cans. They can have a variety of shapes; think of cookie cutters or roof gutters. In addition, cylindrical surfaces need not have a closed cross section such as a circle or triangle, but can be open like a parabola or a hyperbola.

Sketches of three cylindrical surfaces are shown below, one closed and the other two open.
**SUMMARIZE THE MATHEMATICS**

In this investigation, you examined how to generate surfaces in three-dimensional space.

- **a.** Describe two ways that plane curves can be used to generate surfaces.
- **b.** How can you use cross sections parallel to a coordinate plane to identify a surface of revolution?
- **c.** Describe a general procedure to develop an equation for a surface of revolution.
- **d.** What kind of surface of revolution is defined by an equation with only two variables? What is the effect of the omitted variable? How is this seen in intercepts and cross sections at intercepts?
- **e.** How does the generation of a cylindrical surface differ from that of a surface of revolution?

*Be prepared to share your descriptions and thinking with the class.*

**CHECK YOUR UNDERSTANDING**

Consider surfaces that can be generated using an ellipse.

- **a.** Sketch the graph of \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \) in the \( xy \)-plane of a three-dimensional coordinate system.

- **b.** Sketch the surface of revolution generated by rotating \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \) about the \( y \)-axis. Then find the equation of the surface of revolution.

- **c.** Sketch the cylindrical surface with equation \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \).

- **d.** Compare the equations of the surfaces in Parts b and c and summarize the information revealed by the two different forms.
Genie: I would like everyone’s attention because we would like to take a close look at #6 before we finish class today. OK. Are we ready? OK here we have an example of a curve, \( z - y^4 = 0 \), in the yz-plane and we are going to rotate this around the z-axis. OK. Now eventually, we want to be able to write an equation in three-dimensions for \( x, y, \) and \( z \), for the curve that we generate. OK. Why is this one a little more difficult to do than the ones be have been doing previously – in terms of writing the equation? Why is this a little tougher? How was it we were writing the equations before? Maybe I should revisit that. What were we thinking about here when we were writing equations of this? <Pointing to white board and problem #2.> Jack, what were you thinking about?

Jack: We were looking just at the traces.

Genie: At the traces of the cross sections. Right? OK - and knew we were getting some circles and parabolas there. So, what makes this one tougher to do that with? Sophie?

Sophie: It’s not exactly a parabola because it is flatter at the bottom since it is \( y \) to the 4\(^{th} \) and not \( y \) squared.

Genie: it’s not only not exactly a parabola.

Sophie: It’s not.

Student: It’s a U shape.

Genie: It’s like a U shape, right. But not a parabola. Definitely, not. OK

Genie: So, we have another situation here and we’ve got to come up with maybe an alternate strategy for writing our equation for the surface that we are going to generate. OK. Can someone explain to us the thinking that you went through in question 6 in order to justify what the equation was? What was it we... What was the strategy that was developed here to get this equation? Anybody else here? Matt?

Matt: Because you are rotating around a line.

Genie: Speak louder, Matt.

Matt: Because you are rotating the shape around a line trying to get the point that will make that a circle around that line...

Genie: Ah Huh.

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\(^2\) Video clip available at: http://www.wmich.edu/cpmp/classroomglimpses.html
Matt: Um. Which is why. <Inaudible>

Genie: OK, 'cause as we talked about earlier, what is true about any rotation? When you rotate a point around something what happens? What happens when you rotate a point around something? Sophie?

Sophie: Distance is preserved.

Genie: Distance is preserved. OK – So, we are going to get that circle; OK. OK. And it says here: “where r is a function of z.” Why is that true? Why is r a function of z? Ian?

Ian: Because the absolute value of y is equal to r. And the original equation is z equals y to the 4th.

Genie: OK, z…

Ian: So you substitute in r for z.

Genie: OK, let’s slow down. But you’re good. I want to hear this from you. So, start over again, Ian.

Ian: So, z = y^4.

Genie: OK

Ian: Um.. We showed...

Genie: That’s the same as this equation. <Points to whiteboard>

Ian: Yep. We showed that the absolute value of y = r.

Genie: Why does the absolute value of y equal the radius?

Ian: Because on the... on one of the traces, you can see that the distance from the umm z-axis <Genie: ah huh...> to the umm... to the bowl shape

Genie: OK, so it’s more of a bowl shape like this. <Draws on whiteboard>

Ian: Yep, the distance from... <waits for Genie making drawing>

Genie: Ah, huh the distance from what?

Ian: The z-axis to the edge of the bowl shape.

Genie: Is what?

Ian: Is equal to the y-distance.
Genie: Oh it’s the y-coordinate. Isn’t it? OK. Why do we have to say absolute value of y is r? Ian?

Ian: So, it could be on either side of the z-axis.

Genie: OK so it could be on either ... it could be a negative y-coordinate. OK. So, we know the absolute value of y is r. So, where do we go from there, Ian?

Ian: Umm...So, then you can substitute in r for y.

Genie: r for what?

Ian: And say that z = r⁴.

Genie: Oh. z = r⁴. OK. All right. Where can we go from there to help us get our equation? Kate?

Kate: r⁴ is the same as, r² squared.

Genie: r² squared. OK.

Kate: Yah. And since r² = x² + y² <Genie: Ah..> You can square x² + y².

Genie: And up here we have r² = x² + y². So, what does that mean about z? If z is (r²)², Kate, it also has to be what?

Kate: (x² + y²)²

Genie: x² + y² what?

Kate: Squared.

Genie: Squared. And we have our equation for this surface z = (x² + y²)². OK. Did anyone think about this differently? Al?

Al: No. Just if z = (x² + y²)², isn’t that the same as x⁴ + 2x²y² + y⁴?

Genie: Yes. I could multiply it out. And I think back here in this group Katie, you did that. Right. OK. (x² + y²) times itself <writes (x² + y²) (x² + y²) on the whiteboard.> And you could write it in that form if you wanted. x⁴ + 2x²y² + y⁴ OK would be equivalent. That would be another way to write it. But its OK if you leave it in the first form too.

Genie: OK, So, basically what is our strategy here, in general? To do what? Chris?

Chris: Kinda take the traces <inaudible> rotating around the axis